Zeroth-order Stochastic Compositional Algorithms for Risk-Aware Learning

Dionysios S. Kalogerias and Warren B. Powell

Abstract

We present *Free-MESSAGE^p*, the first zeroth-order algorithm for convex mean-semideviation-based risk-aware learning, which is also the first three-level zeroth-order compositional stochastic optimization algorithm, whatsoever. Using a non-trivial extension of Nesterov's classical results on Gaussian smoothing, we develop the *Free-MESSAGE^p* algorithm from first principles, and show that it essentially solves a smoothed surrogate to the original problem, the former being a uniform approximation of the latter, in a useful, convenient sense. We then present a complete analysis of the *Free-MESSAGE^p* algorithm, which establishes convergence in a user-tunable neighborhood of the optimal solutions of the original problem, as well as explicit convergence rates for both convex and strongly convex costs. Orderwise, and for fixed problem parameters, our results demonstrate no sacrifice in convergence speed compared to existing first-order methods, while striking a certain balance among the condition of the problem, its dimensionality, as well as the accuracy of the obtained results, naturally extending previous results in zeroth-order risk-neutral learning.

1 Introduction

Statistical machine learning traditionally deals with the determination and characterization of optimal decision rules minimizing an expected cost criterion, quantifying, for instance, regression or misclassification error in relevant applications, on the basis of available training data [16, 19, 42]. Still, the expected cost paradigm is not appropriate, say, in applications involving *highly dispersive disturbances*, such as heavy tailed, skewed or multimodal noise, or in applications whose purpose is to *imitate uncertain human behavior*. In the first case, merely optimizing the expected cost is often statistically meaningless, since the resulting optimal prediction errors might exhibit unstable or erratic behavior, even with a small expected value. In the second case, as aptly put in [7], the fact is that human decision makers are inherently risk-averse, because they prefer consistent sequences of predictions instead of highly variable ones, even if the latter contain slightly better predictions.

Such situations motivate developments in the area of *risk-aware statistical learning*, in which expectation in the learning objective is replaced by more general functionals, called *risk measures* [38], whose purpose is to effectively quantify the statistical variability of the cost function considered, in addition to mean performance. Indeed, risk-awareness in learning and optimization has already been explored under various problem settings [1, 5, 7, 17, 20, 21, 22, 26, 29, 36, 40, 43, 46, 48], and has proved useful in many important applications, as well [5, 6, 23, 25, 32, 37].

In this paper, we study risk-aware learning problems in which expectation is generalized to the class of *mean-semideviation risk measures* developed in [22]. Specifically, given any complete

Dionysios S. Kalogerias is with the Department of Electrical & Systems Engineering, University of Pennsylvania (e-mail: dionysis@seas.upenn.edu). Warren B. Powell is with the Department of Operations Research & Financial Engineering (ORFE), Princeton University (e-mail: powell@princeton.edu).

probability space $(\Omega, \mathscr{F}, \mathcal{P})$, and a random element $\boldsymbol{W}:\Omega\to\mathbb{R}^M$ on $(\Omega,\mathscr{F})$ modeling abstractly all the uncertainty involved in the learning task, we consider stochastic programs of the form

$$\inf_{\boldsymbol{x}\in\mathcal{X}}\left\{\phi(\boldsymbol{x})\triangleq\mathbb{E}\left\{F(\boldsymbol{x},\boldsymbol{W})\right\}+c\left\|\mathcal{R}\left(F(\boldsymbol{x},\boldsymbol{W})-\mathbb{E}\left\{F(\boldsymbol{x},\boldsymbol{W})\right\}\right)\right\|_{\mathcal{L}_p}\right\},\tag{1}$$

for $c\in[0,1]$ and *order* $p\in[1,2]$, and whereis Borel in its second argument and convex in its first, $F(\cdot,\boldsymbol{W})\in\mathcal{L}_p(\Omega,\mathscr{F},\mathcal{P};\mathbb{R})\triangleq\mathcal{Z}_p$, the set $\mathcal{X}\subseteq\mathbb{R}^N$ is nonempty, closed and convex, and $\mathcal{R}:\mathbb{R}\to\mathbb{R}$ is a *risk regularizer*, or *risk profile* [22], that is, any *convex*, *nonnegative*, *nondecreasing* and *nonexpansive* map. Hereafter, (1) will be called the *base problem*.

The objective ϕ evaluates the mean-semideviation risk measure $\rho(\cdot)\triangleq\mathbb{E}\{\cdot\}+c\|\mathcal{R}((\cdot)-\mathbb{E}\{\cdot\})\|_{\mathcal{L}_p}$ at $F(\cdot,\boldsymbol{W})$, i.e., $\phi(\cdot)\equiv\rho(F(\cdot,\boldsymbol{W}))$ [22]. The functional ρ generalizes the well-known *mean-upper-semideviation* [38], which is recovered by choosing $\mathcal{R}(\cdot)\equiv(\cdot)_+\triangleq\max\{\cdot,0\}$, and is one of the most popular risk-measures in theory and practice [2, 9, 13, 24, 30, 31, 33, 34]. For $c\in[0,1]$, ρ is a *convex risk measure* [22], ([38], Section 6) on $\mathcal{Z}_p$; thus, ϕ in (1) is convex on $\mathbb{R}^N$, as well.

In (1), the expected cost, called the *risk-neutral part* of the objective, is penalized by a *semideviation term*, called the *risk-averse part* of the objective. The latter explicitly quantifies, for each feasible decision, the deviation of the cost relative to its expectation, interpreted as a standardized statistical benchmark. The risk profile $\mathcal{R}$ acts on this central deviation as a weighting function, and its purpose is to reflect the particular risk preferences of the learner. As partially mentioned above, typical choices for $\mathcal{R}$ include the *hockey stick* $(\cdot)_++\eta$, also known as a *Rectified Linear Unit (ReLU)*, as well as its smooth approximations $(1/t)\log(1+\exp(t(\cdot)))+\eta$, with $t>0$, and $\eta\geq 0$. For a constructive characterization of mean-semideviation risk-measures, the reader is referred to [22].

Stochastic subgradient-based recursive optimization of mean-semideviation risk measures was recently considered in [22], where the so-called *MESSAGE^p algorithm* was proposed and analyzed for solving (1). The work of [22] is based on the fact that (1) can be expressed in nested form (see Section 2), and builds on previous results on general compositional stochastic optimization [44, 45].

In this work, we are interested in solving (1) in a *zeroth-order setting*, using exclusively cost function evaluations, in absence of gradient information. Zeroth-order methods have a long history in both deterministic and risk-neutral stochastic optimization [4, 11, 14, 15, 18, 27, 39, 47], and are of particular interest in applications where gradient information is very difficult, or even impossible to obtain, such as training of deep neural networks [8, 41], nonsmooth optimization [28], clinical trials [7], and, more generally, machine learning *in the field*, simulation-based optimization [10, 39], online auctions and search engines [11], and distributed learning [47]. Still, to the best of our knowledge, the development of zeroth-order methods for possibly nonsmooth risk-aware problems such as (1) and, *more generally*, compositional stochastic optimization problems, is completely unexplored. Our contributions are as follows:

- We present *Free-MESSAGE^p*, the first zeroth-order algorithm for solving (1) within a user-specified accuracy, which is also the first three-level zeroth-order compositional stochastic optimization algorithm, whatsoever. The *Free-MESSAGE^p* algorithm requires exactly *four* cost function evaluations per iteration, and is based on finite difference-based inexact gradient approximation, in the spirit of [14, 15, 28]. Using a non-trivial extension of Nesterov's classical results on Gaussian smoothing [28], which we present and discuss, we develop the *Free-MESSAGE^p* algorithm from first principles, and we show that it exactly solves a *smoothed surrogate* to the original problem, the former being a uniform approximation of the latter.

- We present a complete analysis of the *Free-MESSAGE^p* algorithm, establishing path convergence in a user-specified neighborhood of the optimal solutions of (1), as well as explicit convergence rates for both convex and strongly convex costs. Orderwise, and for *fixed* problem parameters, our results demonstrate *no sacrifice in convergence speed* as compared to the fully gradient-based *MESSAGE*^{p} algorithm [22], and explicitly quantify the benefits of strong convexity on problem conditioning, reflected on the derived rates. Further, we develop explicit sample complexity bounds which quantify the inherent dependence of the performance of *Free-MESSAGE*^{p} on both the size of the limiting neighborhood and the decision dimension, N, and naturally extend fundamental prior work on zeroth-order risk-neutral optimization [28].

2 Basic Properties of the Base Problem

First, it will be convenient to express ϕ in *compositional (or nested) form*, as in [22]. By defining *expectation functions* $\varrho : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}_+$, $\boldsymbol{h} : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}$ and $s : \mathbb{R}^N \to \mathbb{R}$ as

$$\varrho(x) \triangleq x^{1/p}, \; g(\boldsymbol{x}, y) \triangleq \mathbb{E}\{(\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^p\} \text{ and } \boldsymbol{h}(\boldsymbol{x}) \triangleq [\boldsymbol{x} \,|\, s(\boldsymbol{x}) \triangleq \mathbb{E}\{F(\boldsymbol{x}, \boldsymbol{W})\}], \qquad (2)$$

respectively, and provided that the involved quantities are well-defined, ϕ may be reexpressed as

$$\phi(\boldsymbol{x}) \equiv s(\boldsymbol{x}) + c\varrho(g(\boldsymbol{h}(\boldsymbol{x}))), \quad \forall \boldsymbol{x} \in \mathcal{X}. \qquad (3)$$

Further, under appropriate conditions, differentiability of ϕ on $\mathcal{X}$ may be guaranteed as follows.

Lemma 1. (Differentiability of ϕ [22]) *Let s and g be differentiable on $\mathcal{X}$ and* $\mathrm{Graph}_{\mathcal{X}}(s)$*, respectively, and let $\mathcal{R} : \mathbb{R} \to \mathbb{R}$ be such that $\{x \in \mathbb{R} \,|\, \mathcal{R}(x) \equiv 0\} \neq \mathbb{R}$. Also, if $p \in (1, 2]$, and with $\kappa_{\mathcal{R}} \triangleq \sup\{x \in \mathbb{R} \,|\, \mathcal{R}(x) \equiv 0\} \in [-\infty, \infty)$, suppose that $\mathcal{P}(F(\boldsymbol{x}, \boldsymbol{W}) - s(\boldsymbol{x}) \leq \kappa_{\mathcal{R}}) < 1$, for all $\boldsymbol{x} \in \mathcal{X}$. Then ϕ is differentiable on $\mathcal{X}$, and its gradient $\nabla\phi : \mathbb{R}^N \to \mathbb{R}^N$ may be expressed as*

$$\nabla\phi(\boldsymbol{x}) \equiv \nabla s(\boldsymbol{x}) + c\nabla\boldsymbol{h}(\boldsymbol{x}) \nabla g(\boldsymbol{h}(\boldsymbol{x})) \nabla\varrho(g(\boldsymbol{h}(\boldsymbol{x}))), \quad \forall \boldsymbol{x} \in \mathcal{X}. \qquad (4)$$

Lemma 1 states *carefully* the obvious: It verifies the composition rule for deriving the gradient of ϕ, properly handling the root ϱ. Also, Lemma 1 is *not* concerned with actually determining $\nabla\boldsymbol{h}$ and ∇g; it just establishes the existence and intrinsically compositional structure of $\nabla\phi$.

3 Gaussian Smoothing and Its Properties

Let $f : \mathbb{R}^N \to \mathbb{R}$ be Borel. Also, for any $\mathbb{R}^N$-valued random element $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$, and for $\mu \geq 0$, consider another Borel function $f_\mu : \mathbb{R}^N \to \mathbb{R}$, defined as $f_\mu(\cdot) \triangleq \mathbb{E}\{f((\cdot) + \mu\boldsymbol{U})\}$, provided that the involved integral is well-defined and finite for all $\boldsymbol{x} \in \mathbb{R}^N$. In many cases, the smoothed function f_μ may be shown to be differentiable on $\mathbb{R}^N$, even if f is not. A wide class of functions satisfying such a property (under some qualification) is that of *Shift-Lipschitz functions*, or *SLipschitz functions*, for short, which are associated with two additional types of functions, which we call *divergences* and *normal remainders*, as introduced below.

Definition 1. (Divergences) A function $\mathsf{D} : \mathbb{R}^N \to \mathbb{R}$ is called a *stationary divergence*, or simply a *divergence*, if and only if $\mathsf{D}(\boldsymbol{u}) \geq 0$, for all $\boldsymbol{u} \in \mathbb{R}^N$, and $\mathsf{D}(\boldsymbol{u}) \equiv 0 \iff \boldsymbol{u} \equiv \boldsymbol{0}$.

Definition 2. (Normal Remainders) A function $\mathsf{T} : \mathbb{R}^{N_o} \times \mathbb{R}^N \to \mathbb{R}$ is called a *normal remainder* on $\mathcal{F} \subseteq \mathbb{R}^{N_o}$ if and only if, for $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$, $\mathbb{E}\{\mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{U})\} \equiv 0$, for all $\boldsymbol{x} \in \mathcal{F}$ and $\mu \geq 0$.

Definition 3. (Shift-Lipschitz Class) A function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is called *Shift-Lipschitz with parameter* $L < \infty$*, relative to a divergence* $\mathsf{D} : \mathbb{R}^N \rightarrow \mathbb{R}$ *and a normal remainder* $\mathsf{T} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, or $(L, \mathsf{D}, \mathsf{T})$-*SLipschitz* for short, *on a subset* $\mathcal{F} \subseteq \mathbb{R}^N$, if and only if, for every $\boldsymbol{u} \in \mathbb{R}^N$,

$$\sup_{\boldsymbol{x} \in \mathcal{F}} |f(\boldsymbol{x} + \boldsymbol{u}) - f(\boldsymbol{x}) - \mathsf{T}(\boldsymbol{x}, \boldsymbol{u})| \leq L\mathsf{D}(\boldsymbol{u}). \tag{5}$$

Apparently, every (real-valued) L-Lipschitz function on $\mathbb{R}^N$, with respect to some norm $\|\cdot\|_* : \mathbb{R}^N \rightarrow \mathbb{R}_+$, is $(L, \|\cdot\|_*, 0)$-SLipschitz on $\mathbb{R}^N$. Similarly, every L-smooth function f on $\mathbb{R}^N$ is $\big(L/2, \|\cdot\|_2^2, (\nabla f(\bullet))^{\boldsymbol{T}}(\cdot)\big)$-SLipschitz on $\mathbb{R}^N$; indeed, f has L-Lipschitz gradient if and only if

$$\left| f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2) - (\nabla f(\boldsymbol{x}_2))^{\boldsymbol{T}} (\boldsymbol{x}_1 - \boldsymbol{x}_2) \right| \leq L \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2^2 /2, \ \forall (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{6}$$

But there are many non-Lipschitz or non-smooth functions, which can be shown to be SLipschitz, at least on some proper subset $\mathcal{F} \subset \mathbb{R}^N$, but where still $\boldsymbol{u} \in \mathbb{R}^N$ (see Definition 3). This is the main reason for working with the SLipschitz class and its extensions, as it provides substantially increased degrees of freedom regarding the choice of the cost function in (1). For two concrete toy examples of non-Lipschitz, non-smooth, but SLipschitz functions, see Appendix A.

We now formulate the next central result, providing several useful properties of f_μ. Simpler versions of this result have been presented earlier in the seminal paper [28], however under more restrictive conditions on f.

Lemma 2. (Properties of f_μ) *Suppose that, for every* $\mu > 0$ *and for every* $0 \leq B < \infty$,

$$\mathbb{E}\{\exp(\|\mu \boldsymbol{U}\|_2 B) \|\mu \boldsymbol{U}\|_2 |f(\mu \boldsymbol{U})|\} < \infty, \quad \boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N). \tag{7}$$

Then, for any subset $\mathcal{F} \subseteq \mathbb{R}^N$*, the following statements are true:*

- *For every* $\mu \geq 0$, f_μ *is well-defined and finite on* $\mathcal{F}$*. Further, if, for some divergence* $\mathsf{D} : \mathbb{R}^N \rightarrow \mathbb{R}$ *and normal remainder* $\mathsf{T} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, f *is* $(L, \mathsf{D}, \mathsf{T})$*-SLipschitz on* $\mathcal{F}$,

$$\sup_{\boldsymbol{x} \in \mathcal{F}} |f_\mu(\boldsymbol{x}) - f(\boldsymbol{x})| \leq L\mathbb{E}\{\mathsf{D}(\mu \boldsymbol{U})\}. \tag{8}$$

- *If* f *is convex on* $\mathbb{R}^N$*, so is* f_μ*, and* f_μ *overestimates* f *everywhere on* $\mathcal{F}$*.*

- *For every* $\mu > 0$, f_μ *is differentiable on* $\mathcal{F}$*, and its gradient* $\nabla f_\mu : \mathbb{R}^N \rightarrow \mathbb{R}^N$ *may be written as*

$$\nabla f_\mu(\boldsymbol{x}) \equiv \mathbb{E}\bigg\{ \frac{f(\boldsymbol{x} + \mu \boldsymbol{U}) - f(\boldsymbol{x})}{\mu} \boldsymbol{U} \bigg\}, \ \forall \boldsymbol{x} \in \mathcal{F}, \tag{9}$$

  where integration is in the sense of Lebesgue. Further, if f *is* $(L, \mathsf{D}, \mathsf{T})$*-SLipschitz on* $\mathcal{F}$*, then*

$$\mathbb{E}\left\{ \left\| \frac{f(\boldsymbol{x} + \mu \boldsymbol{U}) - f(\boldsymbol{x})}{\mu} \boldsymbol{U} \right\|_2^2 \right\} \leq \frac{1}{\mu^2} \mathbb{E}\big\{ \big(L\mathsf{D}(\mu \boldsymbol{U}) + |\mathsf{T}(\boldsymbol{x}, \mu \boldsymbol{U})| \big)^2 \|\boldsymbol{U}\|_2^2 \big\}, \ \forall \boldsymbol{x} \in \mathcal{F}. \tag{10}$$

Proof of Lemma 2. See Appendix B. ■

Driven by Lemma 2, we also introduce a notion of *effectiveness* of a divergence-remainder pair, or (D, T)-*pair*, for short, which quantifies the accuracy of Gaussian smoothing, in general terms.

Definition 4. (Effectiveness of Gaussian Smoothing) Let $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$ and fix $q \geq 2$. Then:

- A (D, T)-pair is called q***-effective*** *on* $\mathcal{F} \subseteq \mathbb{R}^N$ if and only if there are Borel functions $\mathsf{d} : \mathbb{R}^N \to \mathbb{R}$ and $\mathsf{t}_q : \mathcal{F} \times \mathbb{R}^N \to \mathbb{R}$, such that, for some $\varepsilon \geq 0$, $\mu_o \in (0, \infty]$, and for all $\mu \leq \mu_o$,

$$\mathsf{D}(\mu \boldsymbol{u}) \leq \mu^{1+\varepsilon} \mathsf{d}(\boldsymbol{u}) \quad \text{and} \quad \|\mathsf{T}([\boldsymbol{x}, \boldsymbol{Q}], \mu \boldsymbol{u})\|_{\mathcal{L}_q} \leq \mu \mathsf{t}_q(\boldsymbol{x}, \boldsymbol{u}), \quad \forall (\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{F} \times \mathbb{R}^N, \tag{11}$$

  where $\boldsymbol{Q}$ is $\mathscr{F}$-measurable, $\mathsf{d}(\boldsymbol{U}) \in \mathcal{Z}_q$ and $\mathsf{t}_q(\cdot, \boldsymbol{U}) \in \mathcal{Z}_q$.

- A (D, T)-pair is called q***-stable*** *on* $\mathcal{F}$ if and only if it is q-effective on $\mathcal{F}$, with $\mathsf{d}(\boldsymbol{U}) \|\boldsymbol{U}\|_2^{2/\overline{q}} \in \mathcal{Z}_{\overline{q}}$ and $\mathsf{t}_q(\cdot, \boldsymbol{U}) \|\boldsymbol{U}\|_2^{2/\overline{q}} \in \mathcal{Z}_{\overline{q}}$, for all $\overline{q} \in [2, q]$.

- A (D, T)-pair is called ***uniformly*** q_o***-effective (stable)*** *on* $\mathcal{F}$ if and only if it is q-effective (stable) on $\mathcal{F}$ and, further, it holds that $\sup_{\boldsymbol{x} \in \mathcal{F}} \|\mathsf{t}_q(\boldsymbol{x}, \boldsymbol{U})\|_{\mathcal{L}_{\overline{q}}} < \infty$ (plus $\sup_{\boldsymbol{x} \in \mathcal{F}} \|\mathsf{t}_q(\boldsymbol{x}, \boldsymbol{U}) \|\boldsymbol{U}\|_2^{2/\overline{q}}\|_{\mathcal{L}_{\overline{q}}} < \infty$), for all $\overline{q} \in [2, q]$.

In any case of the above, if $\varepsilon > 0$, then D is called an *efficient divergence.*

In the context of Lemma 2, effectiveness of a (D, T)-pair implies that $\mathbb{E}\{\mathsf{D}(\mu \boldsymbol{U})\}$ in (8) decreases *at least linearly in* μ *as* $\mu \to 0$, whereas stability implies that the right of (10) stays *bounded in* μ *as* $\mu \to 0$. If the (D, T)-pair is uniformly 2-stable, then the right-hand side of (9) is also bounded in $\boldsymbol{x}$. Further, if D is an efficient divergence, then $\mathbb{E}\{\mathsf{D}(\mu \boldsymbol{U})\}$ decreases *superlinearly in* μ *as* $\mu \to 0$. The additional conditions imposed by Definition 4 will be relevant shortly.

Typical examples of effective/stable (D, T)-pairs are the one where $\mathsf{D}(\cdot) \equiv \|\cdot\|_2$ and $\mathsf{T} \equiv 0$, associated with the Lipschitz class on $\mathbb{R}^N$, and that where $\mathsf{D}(\cdot) \equiv \|\cdot\|_2^2$ and $\mathsf{T}([\bullet, \star], \cdot) \equiv \mathsf{T}(\bullet, \cdot) \equiv (\nabla f(\bullet))^{\boldsymbol{T}}(\cdot)$ (see above), associated with the smooth class on $\mathbb{R}^N$. For a slightly more elaborate example, see Appendix C.

4 The *Free-MESSAGE*^{p} Algorithm

The basic idea behind the *Free-MESSAGE*^{p} algorithm is to carefully exploit Lemma 2, and replace the gradients involved in expression (4) of Lemma 1 by appropriate smoothed versions, which may be evaluated by exploiting only zeroth-order information. To this end, for $\mu \geq 0$, define functions $g_\mu : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}_+$ and $\boldsymbol{h}_\mu : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}$ and $s_\mu : \mathbb{R}^N \to \mathbb{R}$ as

$$g_\mu(\boldsymbol{x}, y) \triangleq \mathbb{E}\{(\mathcal{R}(F(\boldsymbol{x} + \mu \boldsymbol{U}, \boldsymbol{W}) - (y + \mu U)))^p\}, \quad \text{and} \tag{12}$$

$$\boldsymbol{h}_\mu(\boldsymbol{x}) \triangleq [\boldsymbol{x} \,|\, s_\mu(\boldsymbol{x}) \triangleq \mathbb{E}\{F(\boldsymbol{x} + \mu \boldsymbol{U}, \boldsymbol{W})\}], \tag{13}$$

where $[\boldsymbol{U}^{\boldsymbol{T}}\, U]^{\boldsymbol{T}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_{N+1})$, $[\boldsymbol{U}^{\boldsymbol{T}}\, U]^{\boldsymbol{T}}$ and $\boldsymbol{W}$ are mutually independent, and where, *temporarily*, we implicitly and arbitrarily assume that the involved expectations are well-defined and finite. Then, for $\mu > 0$, we may consider the *μ-smoothed quasi-gradient of ϕ*

$$\widehat{\nabla}_\mu \phi(\boldsymbol{x}) \equiv \nabla s_\mu(\boldsymbol{x}) + c \nabla \boldsymbol{h}_\mu(\boldsymbol{x}) \nabla g_\mu(\boldsymbol{h}_\mu(\boldsymbol{x})) \nabla \varrho(g_\mu(\boldsymbol{h}_\mu(\boldsymbol{x}))), \quad \forall \boldsymbol{x} \in \mathcal{X}, \tag{14}$$

again provided that everything is well-defined and finite. If, further, the conditions of Lemma 2 are fulfilled, *and* with Fubini's permission, it must be true that, for every $\boldsymbol{x} \in \mathcal{X}$,

$$\nabla \boldsymbol{h}_\mu(\boldsymbol{x}) \equiv [\boldsymbol{I}_N \,|\, \nabla s_\mu(\boldsymbol{x})] = \left[\boldsymbol{I}_N \middle| \mathbb{E}\left\{\frac{F(\boldsymbol{x} + \mu \boldsymbol{U}, \boldsymbol{W}) - F(\boldsymbol{x}, \boldsymbol{W})}{\mu} \boldsymbol{U}\right\}\right], \tag{15}$$

Algorithm 1 *Free-MESSAGE^p*

Input: Initial points $\boldsymbol{x}^0 \in \mathcal{X}$, $y^0 \in \mathcal{Y}$, $z^0 \in \mathcal{Z}$, stepsizes $\{\alpha_n\}_n$, $\{\beta_n\}_n$, $\{\gamma_n\}_n$, IID sequences $\{\boldsymbol{W}_1^n\}_n$, $\{\boldsymbol{W}_2^n\}_n$, penalty coefficient $c \in [0,1]$, smoothing parameter μ.
Output: Sequence $\{\boldsymbol{x}^n\}_{n\in\mathbb{N}}$.
1: **for** $n = 0, 1, 2, \ldots$ **do**
2: Sample $\boldsymbol{U}_1^{n+1} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$, and evaluate $F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_1^{n+1}, \boldsymbol{W}_1^{n+1}\big)$.
3: Update (First SA Level):

$$y^{n+1} = \Pi_{\mathcal{Y}}\big\{(1-\beta_n)\, y^n + \beta_n F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_1^{n+1}, \boldsymbol{W}_1^{n+1}\big)\big\}$$

4: Sample $\big[\big(\boldsymbol{U}_2^{n+1}\big)^{\boldsymbol{T}}\, U^{n+1}\big]^{\boldsymbol{T}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_{N+1})$, and evaluate $F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big)$.
5: Update (Second SA Level):

$$z^{n+1} = \begin{cases} 1, & \text{if } p = 1 \\ \Pi_{\mathcal{Z}}\big\{(1-\gamma_n)\, z^n + \gamma_n \big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p\big\}, & \text{if } p > 1 \end{cases}$$

6: Evaluate $F\big(\boldsymbol{x}^n, \boldsymbol{W}_1^{n+1}\big)$ and $F\big(\boldsymbol{x}^n, \boldsymbol{W}_2^{n+1}\big)$.
7: Define auxiliary variables:

$$\Delta_1 = \frac{F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_1^{n+1}, \boldsymbol{W}_1^{n+1}\big) - F\big(\boldsymbol{x}^n, \boldsymbol{W}_1^{n+1}\big)}{\mu}$$
$$\Delta_2 = \frac{\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu\boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - \big(\mathcal{R}\big(F\big(\boldsymbol{x}^n, \boldsymbol{W}_2^{n+1}\big) - y^n\big)\big)^p}{\mu}$$
$$\boldsymbol{\Delta} = (z^n)^{(1-p)/p}\,\big(\boldsymbol{U}_2^{n+1} + \Delta_1 \boldsymbol{U}_1^{n+1} U^{n+1}\big)\Delta_2$$

8: Update (Third SA Level):

$$\boldsymbol{x}^{n+1} = \Pi_{\mathcal{X}}\big\{\boldsymbol{x}^n - \alpha_n\big(\Delta_1 \boldsymbol{U}_1^{n+1} + c\boldsymbol{\Delta}\big)\big\}$$

9: **end for**

and, for every $(\boldsymbol{x}, y) \in \mathrm{Graph}_{\mathcal{X}}(s_\mu)$,

$$\nabla g_\mu(\boldsymbol{x}, y) = \mathbb{E}\left\{\frac{(\mathcal{R}(F(\boldsymbol{x} + \mu\boldsymbol{U}, \boldsymbol{W}) - (y + \mu U)))^p - (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^p}{\mu}\begin{bmatrix}\boldsymbol{U}\\ U\end{bmatrix}\right\}. \tag{16}$$

The quasi-gradient $\widehat{\nabla}_\mu \phi$ suggests a compositional (nested) Stochastic Approximation (SA) scheme for *approximating* a stochastic gradient for ϕ. Similarly to [22, 44, 45], this scheme consists of *three* SA levels and presumes the existence of *two* mutually independent, Independent and Identically Distributed (IID) information streams, $\{\boldsymbol{W}_1^n\}_n$, $\{\boldsymbol{W}_2^n\}_n$, accessible by a $\mathcal{Z}$*eroth*-$\mathcal{O}$*rder* $\mathcal{S}$ampling $\mathcal{O}$racle ($\mathcal{ZOSO}$) for F. We also assume the existence of a *Gaussian sampler*, generating independent standard Gaussian elements on $\mathbb{R}^{N+1}$, mutually independently of $\{\boldsymbol{W}_1^n\}_n$ and $\{\boldsymbol{W}_2^n\}_n$.

The *Free-MESSAGE^p* algorithm is presented in Algorithm 1, where the updates of the first and second SA levels are clearly specified. For the third SA level, given $F\big(\boldsymbol{x}^n, \boldsymbol{W}_1^{n+1}\big)$ and $F\big(\boldsymbol{x}^n, \boldsymbol{W}_2^{n+1}\big)$, and upon defining finite differences $\Delta_1^{n+1} : \mathbb{R}^N \times \Omega \to \mathbb{R}$ and $\Delta_2^{n+1} : \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ as

$$\Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n) \triangleq \frac{F(\boldsymbol{x}^n + \mu \boldsymbol{U}_1^{n+1}, \boldsymbol{W}_1^{n+1}) - F(\boldsymbol{x}^n, \boldsymbol{W}_1^{n+1})}{\mu} \quad \text{and} \tag{17}$$

$$\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}^n, y^n) \triangleq \frac{(\mathcal{R}(F(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}) - \mu U^{n+1} - y^n))^p - (\mathcal{R}(F(\boldsymbol{x}^n, \boldsymbol{W}_2^{n+1}) - y^n))^p}{\mu}, \tag{18}$$

a *stochastic* quasi-gradient $\widehat{\nabla}_\mu^{n+1}\phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is formed as (compare with (14))

$$\begin{aligned}
\widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n, y^n, z^n) &\triangleq \Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1} + c(z^n)^{\frac{1-p}{p}} \left[\boldsymbol{I}_N \middle| \Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1} \right] \Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}^n, y^n) \begin{bmatrix} \boldsymbol{U}_2^{n+1} \\ U^{n+1} \end{bmatrix} \\
&\equiv \Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1} + c(z^n)^{\frac{1-p}{p}} (\boldsymbol{U}_2^{n+1} + \Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1}U^{n+1}) \Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}^n, y^n) \\
&\triangleq \Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1} + c\boldsymbol{\Delta}_{\mu,p}^{n+1}(\boldsymbol{x}^n, y^n, z^n).
\end{aligned} \tag{19}$$

Then, the current estimate $\boldsymbol{x}^n$ is finally updated via a projected quasi-gradient step as

$$\boldsymbol{x}^{n+1} \equiv \Pi_{\mathcal{X}}\{\boldsymbol{x}^n - \alpha_n \widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n, y^n, z^n)\}. \tag{20}$$

5 μ-Smoothed Convex Risk-Averse Surrogates

So far, most mathematical arguments presented in Section 4 have been *imprecise*, since we discussed neither well-definiteness of g_μ, $\boldsymbol{h}_\mu$ and $\widehat{\nabla}_\mu\phi$, nor fulfillment of the conditions of Lemma 2. Here, we resolve all technicalities, and also reveal the actual usefulness of $\widehat{\nabla}_\mu\phi$ in solving problem (1). Our discussion will revolve around the *perturbed cost* $F((\cdot) + \mu\boldsymbol{U}, \boldsymbol{W}) - \mu U \in \mathcal{Z}_p$, ranked via the risk measure ρ. Accordingly, we consider the well-defined, finite-valued function $\phi_\mu : \mathbb{R}^N \to \mathbb{R}$ as

$$\phi_\mu(\boldsymbol{x}) \triangleq \rho([F(\boldsymbol{x} + \mu\boldsymbol{U}, \boldsymbol{W}) - \mu U]), \quad \boldsymbol{x} \in \mathcal{X}. \tag{21}$$

We also impose regularity conditions on the cost F and risk profile $\mathcal{R}$, as follows.

Assumption 1. *F and $\mathcal{R}$ satisfy the following conditions:*

C0 *The functions s and g obey (7).*

C1 *There is $G < \infty$, and a (D, T)-pair, such that*

$$\sup_{\boldsymbol{x} \in \mathcal{X}} \|F(\boldsymbol{x} + \boldsymbol{u}, \boldsymbol{W}) - F(\boldsymbol{x}, \boldsymbol{W}) - \mathsf{T}([\boldsymbol{x}, \boldsymbol{W}], \boldsymbol{u})\|_{\mathcal{L}_2} \le G\mathsf{D}(\boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathbb{R}^N.$$

C2 *There is $V < \infty$, such that $\sup_{\boldsymbol{x} \in \mathcal{X}} \|F(\boldsymbol{x}, \boldsymbol{W})\|_{\mathcal{L}_2} \le V$.*

C3 *The associated (D, T)-pair is uniformly 2-effective on $\mathcal{X}$, and we define $\mathcal{D}_i \triangleq \|\mathsf{d}(\boldsymbol{U})\|_{\mathcal{L}_i}$, for $i \in \{1, 2\}$, and $\mathcal{T}_2 \triangleq \sup_{\boldsymbol{x} \in \mathcal{X}} \|\mathsf{t}_q(\boldsymbol{x}, \boldsymbol{U})\|_{\mathcal{L}_2} < \infty$ (where $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$).*

C4 *If $p \in (1, 2]$, there is $\eta > 0$, such that $\inf_{x \in \mathbb{R}} \mathcal{R}(x) \ge \eta$. Otherwise, $\eta \equiv 0$.*

Under Assumption 1 and exploiting Lemma 2, the next result establishes that ϕ_μ qualifies as a *surrogate* to the base problem (1). Hereafter, let $\mathcal{X}_\mu^o \triangleq \arg\min_{\boldsymbol{x} \in \mathcal{X}} \phi_\mu(\boldsymbol{x})$.

Lemma 3. (Smoothed Convex Surrogates) *Suppose that Assumption 1 is in effect. Then, for every* $0 \le \mu \le \mu_o$, ϕ_μ *is convex and differentiable everywhere on* $\mathcal{X}$, $\nabla\phi_\mu \equiv \widehat{\nabla}_\mu \phi$, *where* $\boldsymbol{h}_\mu$ *and* g_μ *are well-defined, the gradients* $\nabla \boldsymbol{h}_\mu$ *and* ∇g_μ *are given by (15) and (16), respectively, and*

$$\sup_{\boldsymbol{x}\in\mathcal{X}} |\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})| \le \mu^{1+\varepsilon} G\mathcal{D}_1 + c\mathcal{C}(\mu)\left(\mu^{1+\varepsilon}G(\mathcal{D}_1+\mathcal{D}_2) + \mu(\mathcal{T}_2+1)\right), \tag{22}$$

with

$$\mathcal{C}(\mu) \triangleq \mathbb{1}_{\{p\equiv 1\}} + \eta^{-p/2}\big(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon}G(\mathcal{D}_1+\mathcal{D}_2) + \mu(\mathcal{T}_2+1)\big)^{p/2}\mathbb{1}_{\{p\in(1,2]\}}. \tag{23}$$

Additionally, if $\mathcal{X}^o_\mu \ne \emptyset$, *then, for every* $\boldsymbol{x}^o \in \mathcal{X}^o_\mu$ *and* $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{X}\times\mathcal{X}$,

$$\begin{aligned}\phi(\boldsymbol{x}_1) - \phi(\boldsymbol{x}_2) &\le \phi_\mu(\boldsymbol{x}_1) - \phi_\mu(\boldsymbol{x}^o) + 2\sup_{\boldsymbol{x}\in\mathcal{X}}|\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})| \\ &\le \phi_\mu(\boldsymbol{x}_1) - \phi_\mu(\boldsymbol{x}^o) + \Sigma^o\mu(\mu^\varepsilon + c),\end{aligned} \tag{24}$$

where $\Sigma^o \triangleq 2\max\{G\mathcal{D}_1, \mathcal{C}(\mu)(\mu^\varepsilon G(\mathcal{D}_1+\mathcal{D}_2) + (\mathcal{T}_2+1))\}$.

Lemma 3 suggests that ϕ_μ should be useful as a *proxy* for studying *Free-MESSAGE^p* as a method to solve (1). Specifically, inequality (24) is of key importance to the convergence analysis of the *Free-MESSAGE^p* algorithm, discussed later in Section 6. Lemma 3 will be proved in several stages, as follows.

5.1 Proof of Lemma 3

First, an immediate but very useful consequence of Assumption 1 is the following proposition. The proof is elementary and is omitted.

Proposition 1. (Implied Properties of $F(\cdot, \boldsymbol{W})$ I) *Suppose that condition* **C1** *of Assumption 1 is in effect. Then the function* $\mathsf{T}(\bullet,\cdot) \triangleq \mathbb{E}\{\mathsf{T}([\bullet,\boldsymbol{W}],\cdot)\}$ *is a normal remainder on* $\mathcal{X}$. *Further, it is true that, for every* $\boldsymbol{u}\in\mathbb{R}^N$,

$$\begin{aligned}&\sup_{\boldsymbol{x}\in\mathcal{X}} |s(\boldsymbol{x}+\boldsymbol{u}) - s(\boldsymbol{x}) - \mathsf{T}(\boldsymbol{x},\boldsymbol{u})| \\ &\qquad \le \sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\{|F(\boldsymbol{x}+\boldsymbol{u},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W}) - \mathsf{T}([\boldsymbol{x},\boldsymbol{W}],\boldsymbol{u})|\} \le G\mathsf{D}(\boldsymbol{u}),\end{aligned} \tag{25}$$

In other words, $\mathbb{E}\{F(\cdot,\boldsymbol{W})\}$ *is* $(G,\mathsf{D},\mathsf{T})$*-SLipschitz on* $\mathcal{X}$, *and more. If, additionally, condition* **C2** *is in effect, it is true that*

$$|s(\boldsymbol{x}+\boldsymbol{u})| \le \mathbb{E}\{|F(\boldsymbol{x}+\boldsymbol{u},\boldsymbol{W})|\} \le \|F(\boldsymbol{x}+\boldsymbol{u},\boldsymbol{W})\|_{\mathcal{L}_2} \le G\mathsf{D}(\boldsymbol{u}) + \|\mathsf{T}([\boldsymbol{x},\boldsymbol{W}],\boldsymbol{u})\|_{\mathcal{L}_2} + V, \tag{26}$$

for every $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X}\times\mathbb{R}^N$.

For the rest of this section, and by condition **C3**, define the set

$$\mathcal{Y}' \triangleq \left[-V, \mu^{1+\varepsilon}G\mathcal{D}_1 + V\right]. \tag{27}$$

Note that, later in Section 4, we actually set $\mathcal{Y}\equiv\mathcal{Y}'$. Then, leveraging Proposition 1 and Assumption 1, as well as Lemma 2, we have the following result.

Lemma 4. (Existence & Properties of s_μ and g_μ) *Suppose that Assumption 1 is in effect. Then, for some $\varepsilon \geq 0$ and $\mu_o \in (0,\infty]$ according to Definition 4, the following statements are true:*

- *For every $0 \leq \mu \leq \mu_o$, s_μ is well-defined, finite, convex on $\mathcal{X}$, and*

$$\sup_{\boldsymbol{x}\in\mathcal{X}} |s_\mu(\boldsymbol{x}) - s(\boldsymbol{x})| \leq \mu^{1+\varepsilon} G\mathcal{D}_1. \tag{28}$$

  Further, $s_\mu(\boldsymbol{x}) \geq s(\boldsymbol{x})$, for every $\boldsymbol{x} \in \mathcal{X}$.

- *For every $0 < \mu \leq \mu_o$, s_μ is differentiable everywhere on $\mathcal{X}$, and*

$$\nabla s_\mu(\boldsymbol{x}) \equiv \mathbb{E}\left\{\frac{F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W})}{\mu}\boldsymbol{U}\right\}, \tag{29}$$

  for every $\boldsymbol{x} \in \mathcal{X}$. Further,

$$\mathbb{E}\left\{\left\|\frac{F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W})}{\mu}\boldsymbol{U}\right\|_2^2\right\} \leq \mathbb{E}\big\{\big(\mu^{\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\big)^2 \|\boldsymbol{U}\|_2^2\big\}. \tag{30}$$

- *For every $0 \leq \mu$, g_μ is well-defined, finite, convex on $\mathcal{X}\times\mathcal{Y}'$, and $g_\mu(\boldsymbol{x},y) \geq g(\boldsymbol{x},y)$, for every $(\boldsymbol{x},y) \in \mathcal{X}\times\mathcal{Y}'$. Further, if $\mu \leq \mu_o$, then for every $(\boldsymbol{x},y_1,y_2) \in \mathcal{X}\times\mathcal{Y}'\times\mathcal{Y}'$, and every $\big[\boldsymbol{u}^{\boldsymbol{T}}\, u\big]^{\boldsymbol{T}} \in \mathbb{R}^{N+1}$, g satisfies the Lipschitz-like property*

$$|g(\boldsymbol{x}+\mu\boldsymbol{u}, y_1+\mu u) - g(\boldsymbol{x},y_2)| \leq \mathcal{C}(\mu,\boldsymbol{x},\boldsymbol{u})\big(\mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{u}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u}) + \mu|u| + |y_1-y_2|\big), \tag{31}$$

  where

$$\mathcal{C}(\mu,\boldsymbol{x},\boldsymbol{u}) \triangleq \begin{cases} 1, & \text{if } p \equiv 1 \\ p\eta^{(p-2)/2}[\mathcal{R}(0) + 2V & \\ \quad + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{u}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u}) + \mu|u|]^{p/2}, & \text{if } p \in (1,2] \end{cases}. \tag{32}$$

- *For every $0 < \mu \leq \mu_o$, g_μ is differentiable everywhere on $\mathcal{X}\times\mathcal{Y}$ and, for every $(\boldsymbol{x},y) \in \mathcal{X}\times\mathcal{Y}'$,*

$$\nabla g_\mu(\boldsymbol{x},y) \equiv \mathbb{E}\left\{\frac{(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - (y+\mu U)))^p - (\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W}) - y))^p}{\mu}\begin{bmatrix}\boldsymbol{U}\\ U\end{bmatrix}\right\}. \tag{33}$$

Proof of Lemma 4. For the first part of the result, we know from Proposition 1 that the function $s(\cdot) \equiv \mathbb{E}\{F(\cdot,\boldsymbol{W})\}$ is $(G,\mathsf{D},\mathsf{T})$-SLipschitz on $\mathcal{X}$. Then, for $0 \leq \mu \leq \mu_o$, we may call the first part of Lemma 2, which implies that the function $\mathbb{E}\{s((\cdot)+\mu\boldsymbol{U})\} \triangleq s'_\mu(\cdot)$ is well-defined and finite on $\mathcal{X}$, and

$$\sup_{x\in\mathcal{X}} |s'_\mu(\boldsymbol{x}) - s(\boldsymbol{x})| \leq G\mathbb{E}\{\mathsf{D}(\mu\boldsymbol{U})\} \leq \mu^{1+\varepsilon} G\mathbb{E}\{\mathsf{d}(\boldsymbol{U})\}. \tag{34}$$

Additionally, since s is convex on $\mathcal{X}$, so is $\mathbb{E}\{s((\cdot)+\mu\boldsymbol{U})\}$, and the latter overestimates the former. Observe, though, that s'_μ is by definition constructed as an *iterated expectation*, first relative to the distribution of $\boldsymbol{W}$, and then relative to that of $\boldsymbol{U}$, and *not* as an expectation relative to their product measure. Nevertheless, from Proposition 1 and condition **C3** we know that, for every $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X} \times \mathbb{R}^N$,

$$\int |F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{w})| \mathcal{P}_{\boldsymbol{W}}(\mathrm{d}\boldsymbol{w}) \le \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{u}) + \mu \mathsf{t}_2(\boldsymbol{x},\boldsymbol{u}) + V, \tag{35}$$

which in turn implies that, for every $\boldsymbol{x} \in \mathcal{X}$,

$$\int \left[\int |F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{w})| \mathcal{P}_{\boldsymbol{W}}(\mathrm{d}\boldsymbol{w})\right] \mathcal{P}_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}) \le \mu^{1+\varepsilon} G\mathbb{E}\{\mathsf{d}(\boldsymbol{U})\} + \mu\mathbb{E}\{\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\} + V < \infty. \tag{36}$$

Then, by Fubini's Theorem (Corollary 2.6.5 and Theorem 2.6.6 in [3]), it follows that the function $\mathbb{E}\{F((\cdot)+\mu\boldsymbol{U},\boldsymbol{W})\} \equiv s_\mu(\cdot)$ is finite on $\mathcal{X}$, and that

$$\begin{aligned} s'_\mu(\boldsymbol{x}) &\equiv \int \left[\int F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{w}) \mathcal{P}_{\boldsymbol{W}}(\mathrm{d}\boldsymbol{w})\right] \mathcal{P}_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}) \\ &\equiv \int F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{w}) [\mathcal{P}_{\boldsymbol{W}} \times \mathcal{P}_{\boldsymbol{U}}] (\mathrm{d}[\boldsymbol{u},\boldsymbol{w}]) \\ &\equiv s_\mu(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{37}$$

since $\boldsymbol{W}$ and $\boldsymbol{U}$ are statistically independent. A similar procedure may be followed for the second part of the lemma, concerning the expression for the gradient of s_μ. Further, it is true that

$$\begin{aligned} &\mathbb{E}\left\{\left\|\frac{F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W})}{\mu}\boldsymbol{U}\right\|_2^2\right\} \\ &\equiv \frac{1}{\mu^2}\mathbb{E}\{\mathbb{E}\{|F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W})|^2 | \boldsymbol{U}\} \|\boldsymbol{U}\|_2^2\} \\ &\equiv \frac{1}{\mu^2}\mathbb{E}\{\mathbb{E}\{|F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W}) - \mathsf{T}([\boldsymbol{x},\boldsymbol{W}],\mu\boldsymbol{U}) + \mathsf{T}([\boldsymbol{x},\boldsymbol{W}],\mu\boldsymbol{U})|^2 | \boldsymbol{U}\} \|\boldsymbol{U}\|_2^2\} \\ &\le \frac{1}{\mu^2}\mathbb{E}\{(\mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}))^2 \|\boldsymbol{U}\|_2^2\} \\ &\equiv \mathbb{E}\{(\mu^{\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}))^2 \|\boldsymbol{U}\|_2^2\}, \end{aligned} \tag{38}$$

which is what we wanted to show.

For the third part, because g is nonnegative, Fubini's Theorem immediately implies that

$$\mathbb{E}\{g(\boldsymbol{x}+\mu\boldsymbol{U}, y+\mu U)\} \equiv \mathbb{E}\{(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}) - (y+\mu U)))^p\} \equiv g_\mu(\boldsymbol{x},y), \tag{39}$$

for all $(\boldsymbol{x},y) \in \mathcal{X} \times \mathcal{Y}'$, and for every $\mu \ge 0$, where the involved integrals always exist. Then, since g satisfies condition (7) of Lemma 2 by assumption (condition **C0**), it follows that g_μ inherits the respective properties. Next, we show that g is Lipschitz-like, as claimed. Indeed, if $p \equiv 1$, we have, for every $(\boldsymbol{x}, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}' \times \mathcal{Y}'$ and $[\boldsymbol{u}^{\boldsymbol{T}}\ u]^{\boldsymbol{T}} \in \mathbb{R}^{N+1}$,

$$\begin{aligned} |g(\boldsymbol{x}+\mu\boldsymbol{u}, y_1+\mu u) - g(\boldsymbol{x}, y_2)| &\le \mathbb{E}\{|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W}) - (y_1+\mu u)) - \mathcal{R}(F(\boldsymbol{x},\boldsymbol{W}) - y_2)|\} \\ &\le \mathbb{E}\{|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W}) - F(\boldsymbol{x},\boldsymbol{W})|\} + \mu|u| + |y_1 - y_2| \\ &\le \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{u}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u}) + \mu|u| + |y_1 - y_2|, \end{aligned} \tag{40}$$

and we are done. When $p \in (1,2]$, we will exploit another uniform estimate

$$
\begin{aligned}
&\|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y)\|_{\mathcal{L}_p}\\
&\quad\le\|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y)\|_{\mathcal{L}_2}\\
&\quad\le\|\mathcal{R}(0)+|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y|\|_{\mathcal{L}_2}\\
&\quad\le\mathcal{R}(0)+|y|+\mu|u|+\|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})\|_{\mathcal{L}_2}\\
&\quad\le\mathcal{R}(0)+2V+\mu^{1+\varepsilon}G\mathcal{D}_1+\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u})+\mu|u|,
\end{aligned}
\tag{41}
$$

which holds everywhere on $\mathcal{X}\times\mathcal{Y}'\times\mathbb{R}^N\times\mathbb{R}$. Similarly,

$$
\|\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y)\|_{\mathcal{L}_p}\le\mathcal{R}(0)+\mu^{1+\varepsilon}G\mathcal{D}_1+2V,
\tag{42}
$$

everywhere on $\mathcal{X}\times\mathcal{Y}'$. Then, for every $(\boldsymbol{x},y_1,y_2)\in\mathcal{X}\times\mathcal{Y}'\times\mathcal{Y}'$, and for every $[\boldsymbol{u}^{\boldsymbol{T}}\,u]^{\boldsymbol{T}}\in\mathbb{R}^{N+1}$, we may write (recall Assumption 1)

$$
\begin{aligned}
&|g(\boldsymbol{x}+\mu\boldsymbol{u},y_1+\mu u)-g(\boldsymbol{x},y_2)|\\
&\quad\le\mathbb{E}\{|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^p-(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^p|\}\\
&\quad\equiv\mathbb{E}\big\{\big|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{2p/2}-(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^{2p/2}\big|\big\}\\
&\quad\equiv\mathbb{E}\big\{\big|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{p/2}-(\mathcal{R}(F(\boldsymbol{x}_2,\boldsymbol{W})-y_2))^{p/2}\big|\\
&\qquad\qquad\times\big((\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{p/2}+(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^{p/2}\big)\big\}\\
&\quad\le\frac{p\eta^{(p-2)/2}}{2}\mathbb{E}\big\{|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1)-\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2)|\\
&\qquad\qquad\times\big((\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{p/2}+(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^{p/2}\big)\big\}\\
&\quad\le\frac{p\eta^{(p-2)/2}}{2}\mathbb{E}\big\{(|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})|+\mu|u|+|y_1-y_2|)\\
&\qquad\qquad\times\big((\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{p/2}+(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^{p/2}\big)\big\}\\
&\quad\le\frac{p\eta^{(p-2)/2}}{2}\big(\|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})\|_{\mathcal{L}_2}+\mu|u|+|y_1-y_2|\big)\\
&\qquad\qquad\times\big(\|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))^{p/2}\|_{\mathcal{L}_2}+\|(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))^{p/2}\|_{\mathcal{L}_2}\big)\\
&\quad\equiv\frac{p\eta^{(p-2)/2}}{2}\big(\|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})\|_{\mathcal{L}_2}+\mu|u|+|y_1-y_2|\big)\\
&\qquad\qquad\times\big(\|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-\mu u-y_1))\|_{\mathcal{L}_p}^{p/2}+\|(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y_2))\|_{\mathcal{L}_p}^{p/2}\big)\\
&\quad\le p\eta^{(p-2)/2}\big(\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u})+\mu|u|+|y_1-y_2|\big)\\
&\qquad\qquad\times\big[\mathcal{R}(0)+2V+\mu^{1+\varepsilon}G\mathcal{D}_1+\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u})+\mu|u|\big]^{p/2}.
\end{aligned}
\tag{43}
$$

Finally, the last part of Lemma 4 may be verified by another application of Fubini's Theorem, as in the first and second part discussed above, or the tower property, and another application of Lemma 2. Enough said. ■

We now prove Lemma 3 for $p\in(1,2]$; the case where $p\equiv 1$ is similar, albeit simpler. To start, for $0\le\mu\le\mu_o$, convexity of ϕ_μ on $\mathcal{X}$ follows from convexity of $F((\cdot)+\mu\boldsymbol{U},\boldsymbol{W})-\mu U$ on $\mathcal{X}$, which may be shown trivially, based on the convexity of $F(\cdot, \boldsymbol{W})$. Next, to verify differentiability of ϕ_μ, it suffices to check the sufficient conditions of Lemma 1. Indeed, since, by condition **C4**, $\inf_{x\in\mathbb{R}} \mathcal{R}(x) \geq \eta > 0$, it is true that $\kappa_{\mathcal{R}} \equiv -\infty$ and, thus, for every $\boldsymbol{x} \in \mathcal{X}$,

$$\mathcal{P}\left(F\left(\boldsymbol{x}+\mu\boldsymbol{U}, \boldsymbol{W}\right) - \mu U - \mathbb{E}\left\{F\left(\boldsymbol{x}+\mu\boldsymbol{U}, \boldsymbol{W}\right) - \mu U\right\} \leq \kappa_{\mathcal{R}}\right) \equiv 0 < 1. \tag{44}$$

Then, Lemma 1 implies that ϕ_μ is differentiable everywhere on $\mathcal{X}$, and also that $\nabla\phi_\mu(\boldsymbol{x}) \equiv \widehat{\nabla}_\mu \phi(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{X}$, which may easily shown by application of the composition rule to ϕ_μ, for which it is true that

$$\begin{aligned}
\phi_\mu(\boldsymbol{x}) &\equiv \mathbb{E}\left\{F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)-\mu U\right\}+c\left\|\mathcal{R}\left(F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)-\mu U-\mathbb{E}\left\{F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)-\mu U\right\}\right)\right\|_{\mathcal{L}_p} \\
&\equiv \mathbb{E}\left\{F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)\right\}+c\left\|\mathcal{R}\left(F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)-\left(\mathbb{E}\left\{F\left(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W}\right)\right\}+\mu U\right)\right)\right\|_{\mathcal{L}_p} \\
&\equiv s_\mu(\boldsymbol{x})+c\varrho(g_\mu(\boldsymbol{h}_\mu(\boldsymbol{x}))), \quad \forall \boldsymbol{x} \in \mathcal{X}.
\end{aligned} \tag{45}$$

Now, because of the fact that (see, for instance, Lemma 4)

$$-V < \inf_{\boldsymbol{x}\in\mathcal{X}} s_\mu(\boldsymbol{x}) \leq \sup_{\boldsymbol{x}\in\mathcal{X}} s_\mu(\boldsymbol{x}) \leq \mu^{1+\varepsilon} G\mathcal{D}_1 + V \iff s_\mu(\boldsymbol{x}) \in \mathcal{Y}', \quad \forall \boldsymbol{x} \in \mathcal{X}, \tag{46}$$

we may invoke Lemma 4, yielding, for every $\boldsymbol{x} \in \mathcal{X}$,

$$\begin{aligned}
&\left|\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})\right| \\
&\quad\leq \left|s_\mu(\boldsymbol{x}) - s(\boldsymbol{x})\right| + c\left|\varrho(g_\mu(\boldsymbol{h}_\mu(\boldsymbol{x}))) - \varrho(g(\boldsymbol{h}(\boldsymbol{x})))\right| \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + c\left|\varrho(g_\mu(\boldsymbol{h}_\mu(\boldsymbol{x}))) - \varrho(g(\boldsymbol{h}(\boldsymbol{x})))\right| \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\left|\mathbb{E}\{g(\boldsymbol{x}+\mu\boldsymbol{U}, s_\mu(\boldsymbol{x})+\mu U)\} - \mathbb{E}\left\{g\left(\boldsymbol{x}, s(\boldsymbol{x})\right)\right\}\right| \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\mathbb{E}\{\left|g\left(\boldsymbol{x}+\mu\boldsymbol{U}, s_\mu(\boldsymbol{x})+\mu U\right) - g\left(\boldsymbol{x}, s(\boldsymbol{x})\right)\right|\} \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\mathbb{E}\{\mathcal{C}\left(\mu, \boldsymbol{x}, \boldsymbol{U}\right)\left(\left|s_\mu(\boldsymbol{x}) - s(\boldsymbol{x})\right| + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + \mu|U|\right)\} \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\mathbb{E}\{\mathcal{C}\left(\mu, \boldsymbol{x}, \boldsymbol{U}\right)\left(\mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + \mu|U|\right)\} \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\left\|\mathcal{C}\left(\mu, \boldsymbol{x}, \boldsymbol{U}\right)\right\|_{\mathcal{L}_2}\left\|\mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + \mu|U|\right\|_{\mathcal{L}_2} \\
&\quad\leq \mu^{1+\varepsilon} G\mathcal{D}_1 + cp^{-1}\eta^{1-p}\left\|\mathcal{C}\left(\mu, \boldsymbol{x}, \boldsymbol{U}\right)\right\|_{\mathcal{L}_2}(\mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathcal{D}_2 + \mu\mathcal{T}_2 + \mu).
\end{aligned} \tag{47}$$

Additionally, it is also true that

$$\begin{aligned}
&\left\|\mathcal{C}\left(\mu, \boldsymbol{x}, \boldsymbol{U}\right)\right\|_{\mathcal{L}_2} \\
&\quad\equiv p\eta^{(p-2)/2}\left\|\left[\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + \mu|u|\right]^{p/2}\right\|_{\mathcal{L}_2} \\
&\quad\equiv p\eta^{(p-2)/2}\left\|\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + \mu|U|\right\|_{\mathcal{L}_p}^{p/2} \\
&\quad\leq p\eta^{(p-2)/2}\left(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\|\mathsf{d}(\boldsymbol{U})\|_{\mathcal{L}_p} + \mu\|\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\|_{\mathcal{L}_p} + \mu\|U\|_{\mathcal{L}_p}\right)^{p/2} \\
&\quad\leq p\eta^{(p-2)/2}\left(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathcal{D}_2 + \mu\mathcal{T}_2 + \mu\right)^{p/2}
\end{aligned} \tag{48}$$

Therefore, for every $\boldsymbol{x} \in \mathcal{X}$, (47) may be further bounded from above as

$$\left|\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})\right| \leq \mu^{1+\varepsilon} G\mathcal{D}_1 + c\eta^{-p/2}\left(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathcal{D}_2 + \mu\mathcal{T}_2 + \mu\right)^{p/2}$$

$$\times\left(\mu^{1+\varepsilon} G \mathcal{D}_1+\mu^{1+\varepsilon} G \mathcal{D}_2+\mu \mathcal{T}_2+\mu\right), \tag{49}$$

and we are done. Finally, if $\mathcal{X}_\mu^o \neq \emptyset$ and $\boldsymbol{x}^o \in \mathcal{X}_\mu^o$, and for every $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{X} \times \mathcal{X}$, we may write

$$\begin{aligned}
\phi(\boldsymbol{x}_1)-\phi(\boldsymbol{x}_2) &\equiv \phi(\boldsymbol{x}_1)-\phi_\mu(\boldsymbol{x}_1)+\phi_\mu(\boldsymbol{x}_1)-\phi_\mu(\boldsymbol{x}^o)+\phi_\mu(\boldsymbol{x}^o)-\phi_\mu(\boldsymbol{x}_2)+\phi_\mu(\boldsymbol{x}_2)-\phi(\boldsymbol{x}_2) \\
&\leq \phi(\boldsymbol{x}_1)-\phi_\mu(\boldsymbol{x}_1)+\phi_\mu(\boldsymbol{x}_1)-\phi_\mu(\boldsymbol{x}^o)+\phi_\mu(\boldsymbol{x}_2)-\phi(\boldsymbol{x}_2) \\
&\leq \phi_\mu(\boldsymbol{x}_1)-\phi_\mu(\boldsymbol{x}^o)+2 \sup_{\boldsymbol{x} \in \mathcal{X}}|\phi_\mu(\boldsymbol{x})-\phi(\boldsymbol{x})|,
\end{aligned} \tag{50}$$

where we have used the fact that $\phi_\mu(\boldsymbol{x}^o) \leq \phi_\mu(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{X}$. ■

6 Convergence Analysis

By Lemma 3, it follows that the compositional quasi-gradient $\widehat{\nabla}_\mu \phi$ (see (14)) is actually the gradient of the function ϕ_μ. Therefore, the *Free-MESSAGE^p* algorithm may be legitimately seen as a zeroth-order method to solve *exactly* the convex mean-semideviation problem

$$\inf_{\boldsymbol{x} \in \mathcal{X}}\left\{\phi_\mu(\boldsymbol{x}) \equiv \rho\left([F(\boldsymbol{x}+\mu \boldsymbol{U}, \boldsymbol{W})-\mu U]\right)\right\}, \tag{51}$$

where $\mu>0$ (if $\mu \equiv 0$, then $\phi_0 \equiv \phi$, and the situation is trivial). Lemma 3 explicitly quantifies the quality of the approximation of ϕ by ϕ_μ, as well. Consequently, it makes sense to *first* study the *Free-MESSAGE^p* algorithm as a method for solving the surrogate (51), and *then* attempt to relate the obtained results to the original problem, using Lemma 3. Our results follow this path. The behavior of the *Free-MESSAGE^p* algorithm will be characterized under the following conditions, extending Assumption 1 of the previous section.

Assumption 2. *Assumption 1 is in effect and conditions* **C1**-**C3** *are strengthened as follows:*

$\overline{\textbf{C1}}$ *There is $G<\infty$, and a (D, T)-pair, as in condition* **C1**, *such that*

$$\sup_{\boldsymbol{x} \in \mathcal{X}}\|F(\boldsymbol{x}+\boldsymbol{u}, \boldsymbol{W})-F(\boldsymbol{x}, \boldsymbol{W})-\mathsf{T}([\boldsymbol{x}, \boldsymbol{W}], \boldsymbol{u})\|_{\mathcal{L}_{4-2 \mathbb{1}_{\{p \equiv 1\}}}} \leq G \mathsf{D}(\boldsymbol{u}), \ \forall \boldsymbol{u} \in \mathbb{R}^N.$$

$\overline{\textbf{C2}}$ *There is $V_p<\infty$, such that $\sup_{\boldsymbol{x} \in \mathcal{X}}\|F(\boldsymbol{x}, \boldsymbol{W})\|_{\mathcal{L}_{2p}} \leq V_p$. Thus, $V_1 \equiv V$.*

$\overline{\textbf{C3}}$ *The associated (D, T)-pair is uniformly $(4-2 \mathbb{1}_{\{p \equiv 1\}})$-stable on $\mathcal{X}$.*

Additionally:

C5 *The sets $\mathcal{Y}$ and $\mathcal{Z}$ are chosen as*

$$\mathcal{Y} \triangleq\left[-V, \mu^{1+\varepsilon} G \mathcal{D}_1+V\right] \text{ and } \mathcal{Z} \triangleq\left[\eta^p,\left(\mathcal{R}(0)+2 V+\mu^{1+\varepsilon} G(\mathcal{D}_1+\mathcal{D}_2)+\mu(\mathcal{T}_2+1)\right)^p\right).$$

C6 *There is $L<\infty$, such that g_μ satisfies the marginal smoothness condition*

$$\sup_{\boldsymbol{x} \in \mathcal{X}}\|\nabla g_\mu(\boldsymbol{x}, y_1)-\nabla g_\mu(\boldsymbol{x}, y_2)\|_2 \leq L|y_1-y_2|, \quad \forall(y_1, y_2) \in \mathcal{Y} \times \mathcal{Y}.$$

Note that condition **C5** of Assumption 2 can be satisfied under various common circumstances, in particular when g is L-smooth globally on $\mathbb{R}^N \times \mathbb{R}$. Note, though, that condition **C5** is significantly weaker than demanding L-smoothness of g.

6.1 Main Implications of Assumption 2

As in the case of Assumption 1, an immediate consequence of Assumption 2 is the following proposition. The proof is omitted.

Proposition 2. (Implied Properties of $F(\cdot, \boldsymbol{W})$ II) *Suppose that conditions* $\overline{\mathbf{C1}}$ *and* $\overline{\mathbf{C2}}$ *of Assumption 2 are in effect. Then, it is true that*

$$\|F(\boldsymbol{x}+\boldsymbol{u}, \boldsymbol{W})\|_{\mathcal{L}_{2p}} \leq G\mathsf{D}(\boldsymbol{u}) + \|\mathsf{T}([\boldsymbol{x}, \boldsymbol{W}], \boldsymbol{u})\|_{\mathcal{L}_{2p}} + V_p, \tag{52}$$

for every $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{X} \times \mathbb{R}^N$. *Further, if condition* $\overline{\mathbf{C3}}$ *is in effect, then, for every* $\mu \in (0, \mu_o]$,

$$\sup_{\boldsymbol{x}\in\mathcal{X}} \|F(\boldsymbol{x}+\mu\boldsymbol{U}, \boldsymbol{W})\|_{\mathcal{L}_{2p}} \leq V_p' \triangleq \mu^{1+\varepsilon} G \|\mathsf{d}(\boldsymbol{U})\|_{\mathcal{L}_{2p}} + \mu \sup_{\boldsymbol{x}\in\mathcal{X}} \|\mathsf{t}_{2p}(\boldsymbol{x}, \boldsymbol{U})\|_{\mathcal{L}_{2p}} + V_p. \tag{53}$$

The main purpose of Assumption 2 is to guarantee boundedness of the gradients appearing in the *Free-MESSAGE^p* algorithm in a certain sense, *uniformly* on the respective feasible sets. In this respect, we have the next result.

Lemma 5. (Gradient Boundedness) *Suppose that Assumption 2 is in effect. Then, for every* $0 < \mu \leq \mu_o$, *there exist problem dependent constants* $B_1 \equiv B_1^\mu < \infty$ *and* $B_2 \equiv B_2^\mu < \infty$, *both increasing and bounded in* μ, *such that*

$$B_1 \geq \sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\left\{\left\|\frac{F(\boldsymbol{x}+\mu\boldsymbol{U}, \boldsymbol{W}) - F(\boldsymbol{x}, \boldsymbol{W})}{\mu}\boldsymbol{U}\right\|_2^2\right\} \quad \text{and} \tag{54}$$

$$B_2 \geq \sup_{(\boldsymbol{x},y)\in\mathcal{X}\times\mathcal{Y}} \mathbb{E}\left\{\left\|\frac{(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{U}, \boldsymbol{W}) - (y+\mu U)))^p - (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^p}{\mu}\begin{bmatrix}\boldsymbol{U}\\ U\end{bmatrix}\right\|_2^2\right\}. \tag{55}$$

Consequently, it follows that $\sup_{\boldsymbol{x}\in\mathcal{X}} \|\nabla s_\mu(\boldsymbol{x})\|_2^2 \leq B_1$ *and* $\sup_{(\boldsymbol{x},y)\in\mathcal{X}\times\mathcal{Y}} \|\nabla g_\mu(\boldsymbol{x}, y)\|_2^2 \leq B_2$, *implying that both* s_μ *and* g_μ *are Lipschitz in the usual sense on* $\mathcal{X}$ *and* $\mathcal{X}\times\mathcal{Y}$, *respectively.*

Proof of Lemma 5. We work assuming that $p \in (1, 2]$. If $p \equiv 1$, the proof follows accordingly. Since (54) follows trivially from Lemma 4, we focus exclusively on showing (55). First, for every pair $(\boldsymbol{x}, y) \in \mathcal{X} \times \mathcal{Y}$, we may carefully write

$$\begin{aligned}
&\mathbb{E}\{|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^p - (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^p|^2\} \\
&\quad \equiv \mathbb{E}\{|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^{2p/2} - (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^{2p/2}|^2\} \\
&\quad \equiv \mathbb{E}\{|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^{p/2} - (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^{p/2}|^2 \\
&\qquad\quad \times |(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^{p/2} + (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^{p/2}|^2\} \\
&\quad \leq \frac{p^2\eta^{(p-2)}}{4}\mathbb{E}\{|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)) - \mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y)|^2 \\
&\qquad\quad \times |(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^{p/2} + (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^{p/2}|^2\} \\
&\quad \leq \frac{p^2\eta^{(p-2)}}{4}\mathbb{E}\{(|F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - F(\boldsymbol{x}, \boldsymbol{W})| + \mu|u|)^2 \\
&\qquad\quad \times |(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u}, \boldsymbol{W}) - (y+\mu u)))^{p/2} + (\mathcal{R}(F(\boldsymbol{x}, \boldsymbol{W}) - y))^{p/2}|^2\}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{p^2\eta^{(p-2)}}{2}\|(|F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})|+\mu|u|)^2\|_{\mathcal{L}_2}\\
&\qquad\times\left(\|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-(y+\mu u)))^p\|_{\mathcal{L}_2}+\|(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y))^p\|_{\mathcal{L}_2}\right)\\
&\equiv \frac{p^2\eta^{(p-2)}}{2}\||F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})|+\mu|u|\|_{\mathcal{L}_4}^2\\
&\qquad\times\left(\|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-(y+\mu u))\|_{\mathcal{L}_{2p}}^p+\|\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y)\|_{\mathcal{L}_{2p}}^p\right)\\
&\le \frac{p^2\eta^{(p-2)}}{2}\left(\||F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})|\|_{\mathcal{L}_4}+\mu|u|\right)^2\\
&\qquad\times\left(\|\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{u},\boldsymbol{W})-(y+\mu u))\|_{\mathcal{L}_{2p}}^p+\|\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y)\|_{\mathcal{L}_{2p}}^p\right)\\
&\le p^2\eta^{(p-2)}\left(\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+\mu|u|\right)^2\\
&\qquad\times\left(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1+\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+\mu|u|\right)^p\\
&\le p^2\eta^{(p-2)}2^{p-1}\left(\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+\mu|u|\right)^2\\
&\qquad\times\left(\left(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1\right)^p+\left(\mu^{1+\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mu\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+\mu|u|\right)^p\right)\\
&\equiv \mu^2p^2\eta^{(p-2)}2^{p-1}\Big(\mu^p\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+|u|\right)^{p+2}\\
&\qquad+\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{u})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u})+|u|\right)^2\left(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1\right)^p\Big).
\end{aligned}
\tag{56}
$$

Therefore, the tower property implies that

$$
\begin{aligned}
&\mathbb{E}\left\{\left\|\frac{(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W})-(y+\mu U)))^p-(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y))^p}{\mu}\begin{bmatrix}\boldsymbol{U}\\U\end{bmatrix}\right\|_2^2\right\}\\
&\equiv\frac{1}{\mu^2}\mathbb{E}\left\{\mathbb{E}\{|(\mathcal{R}(F(\boldsymbol{x}+\mu\boldsymbol{U},\boldsymbol{W})-(y+\mu U)))^p-(\mathcal{R}(F(\boldsymbol{x},\boldsymbol{W})-y))^p|^2|\boldsymbol{U},U\}\left\|\begin{bmatrix}\boldsymbol{U}\\U\end{bmatrix}\right\|_2^2\right\}\\
&\le p^2\eta^{(p-2)}2^{p-1}\Big(\mu^p\mathbb{E}\{\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\right)^{p+2}(\|\boldsymbol{U}\|_2^2+U^2)\}\\
&\quad+\left(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1\right)^p\mathbb{E}\{\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\right)^2(\|\boldsymbol{U}\|_2^2+U^2)\}\Big),
\end{aligned}
\tag{57}
$$

for all $\boldsymbol{x}\in\mathcal{X}$. The proof is now complete, but let us consider the two expectations on the right-hand side of (57) separately, as a (in)sanity check. For the first one, we may write

$$
\begin{aligned}
&\mathbb{E}\{\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\right)^{p+2}(\|\boldsymbol{U}\|_2^2+U^2)\}\\
&\quad\le 2^{p+1}\mathbb{E}\{\left(\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\right)^{p+2}+|U|^{p+2}\right)(\|\boldsymbol{U}\|_2^2+U^2)\}\\
&\quad\equiv 2^{p+1}\Big(\mathbb{E}\{\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\right)^{p+2}\|\boldsymbol{U}\|_2^2\}+\mathbb{E}\{\left(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\right)^{p+2}|U|^{p+2}\}\\
&\qquad+\mathbb{E}\{|U|^{p+2}\|\boldsymbol{U}\|_2^2\}+\mathbb{E}\{|U|^{p+4}\}\Big)\\
&\quad\le 2^{p+1}\Big(2^{p+1}\Big(\mu^{\varepsilon(p+2)}\mathbb{E}\{\left(G\mathsf{d}(\boldsymbol{U})\right)^{p+2}\|\boldsymbol{U}\|_2^2\}+\sup_{\boldsymbol{x}\in\mathcal{X}}\mathbb{E}\{\left(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\right)^{p+2}\|\boldsymbol{U}\|_2^2\}\Big)\\
&\qquad+2^{p+1}\Big(\mu^{\varepsilon(p+2)}\mathbb{E}\{\left(G\mathsf{d}(\boldsymbol{U})\right)^{p+2}\}\mathbb{E}\{|U|^{p+2}\}+\sup_{\boldsymbol{x}\in\mathcal{X}}\mathbb{E}\{\left(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\right)^{p+2}\}\mathbb{E}\{|U|^{p+2}\}\Big)\\
&\qquad+\mathbb{E}\{|U|^{p+2}\}\mathbb{E}\{\|\boldsymbol{U}\|_2^2\}+\mathbb{E}\{|U|^{p+4}\}\Big)\equiv\mathcal{O}(1).
\end{aligned}
\tag{58}
$$

For the second one, the situation is similar. Enough said. ■

6.2 Recursions

We follow the approach taken previously in ([22], Section 4.4), but with appropriate technical modifications in the proofs of the corresponding results, reflecting the problem setting and assumptions considered herein. Because the proof ideas are similar to ([22], Section 4.4), we postpone all proofs of this section to Appendix E. Still, we emphasize that the results presented below crucially exploit gradient boundedness ensured by Lemma 5, which follows as a result of Assumption 2.

Hereafter, let $\{\mathscr{D}^n \subseteq \mathscr{F}\}_{n\in\mathbb{N}}$ be the filtration generated from *all data observed so far, by both the user and the $\mathcal{ZOSO}$*, with $\mathscr{D}^n \triangleq \sigma\{\boldsymbol{x}^i, y^i, z^i, \boldsymbol{W}_1^i, \boldsymbol{W}_2^i, \boldsymbol{U}_1^i, \boldsymbol{U}_2^i, U^i, \forall i \in \mathbb{N}_n\}$, $n \in \mathbb{N}$. Also, if $\mathscr{C}$ is a sub σ-algebra of $\mathscr{F}$, we compactly write $\mathbb{E}\{\cdot\,|\mathscr{C}\} \equiv \mathbb{E}_{\mathscr{C}}\{\cdot\}$. Our first basic result follows.

Lemma 6. (Iterate Increment Growth) *Suppose that Assumption 2 is in effect. Then, for every $0 < \mu \le \mu_o$, there exists a problem dependent constant $\Sigma_p^1 < \infty$, increasing and bounded in μ, such that the process $\{\boldsymbol{x}^n\}_{n\in\mathbb{N}}$ generated by the Free-MESSAGE^p algorithm satisfies the inequality*

$$\mathbb{E}_{\mathscr{D}^n}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\|_2^2\} \le \Sigma_p^1 \alpha_n^2, \tag{59}$$

for all $n \in \mathbb{N}$, almost everywhere relative to $\mathcal{P}$.

Proof of Lemma 6. See Appendix E.1. ■

Using Lemma 6, we have the next result on the growth of the difference $|y^n - s_\mu(\boldsymbol{x}^n)|^2$.

Lemma 7. (First Zeroth-order SA Level: Error Growth) *Suppose that Assumption 2 is in effect. Also, let $\beta_n \in (0,1]$, for all $n \in \mathbb{N}$. Then, for every $0 < \mu \le \mu_o$, there exists a problem dependent constant $\Sigma_p^2 < \infty$, increasing and bounded in μ, such that the process $\{(\boldsymbol{x}^n, y^n)\}_{n\in\mathbb{N}}$ generated by the Free-MESSAGE^p algorithm satisfies the inequality*

$$\mathbb{E}_{\mathscr{D}^n}\{|y^{n+1} - s_\mu(\boldsymbol{x}^{n+1})|^2\} \le (1-\beta_n)|y^n - s_\mu(\boldsymbol{x}^n)|^2 + \Sigma_p^2(\beta_n^2 + \beta_n^{-1}\alpha_n^2), \tag{60}$$

for all $n \in \mathbb{N}$, almost everywhere relative to $\mathcal{P}$.

Proof of Lemma 7. See Appendix E.2. ■

Similarly, when $p > 1$, the growth of $z^n - g_\mu(\boldsymbol{x}^n, y^n)$ may be characterized as follows.

Lemma 8. (Second Zeroth-order SA Level: Error Growth) *Suppose that Assumption 2 is in effect. Also, choose $p > 1$, and let $\beta_n \in (0,1]$, $\gamma_n \in (0,1]$, for all $n \in \mathbb{N}$. Then, for every $0 < \mu \le \mu_o$, there exists a problem dependent constant $\Sigma_p^3 < \infty$, increasing and bounded in μ, such that the process $\{(\boldsymbol{x}^n, y^n, z^n)\}_{n\in\mathbb{N}}$ generated by the Free-MESSAGE^p algorithm satisfies the inequality*

$$\mathbb{E}_{\mathscr{D}^n}\{|z^{n+1} - g_\mu(\boldsymbol{x}^{n+1}, y^{n+1})|^2\} \le (1-\gamma_n)|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2 + \Sigma_p^3(\gamma_n^2 + \gamma_n^{-1}\alpha_n^2 + \gamma_n^{-1}\beta_n^2), \tag{61}$$

for all $n \in \mathbb{N}$, almost everywhere relative to $\mathcal{P}$.

Proof of Lemma 8. See Appendix E.3. ■

Next, let us define a Borel function $g'_\mu : \mathbb{R}^N \to \mathbb{R}$ as

$$g'_\mu(\boldsymbol{x}) \triangleq \mathbb{E}\{(\mathcal{R}(F(\boldsymbol{x} + \mu\boldsymbol{U}, \boldsymbol{W}) - \mu U - s_\mu(\boldsymbol{x})))^p\} \equiv g_\mu(\boldsymbol{x}, s_\mu(\boldsymbol{x})). \tag{62}$$

Also note that, as in the original *MESSAGE*^{p} algorithm [22], it is true that, for every $(n, \boldsymbol{x}) \in \mathbb{N}^+ \times \mathcal{X}$,

$$\mathbb{E}\{\widehat{\nabla}_{\mu}^{n+1}\phi(\boldsymbol{x}, s_{\mu}(\boldsymbol{x}), g'_{\mu}(\boldsymbol{x}))\} \equiv \widehat{\nabla}_{\mu}\phi(\boldsymbol{x}), \tag{63}$$

implying that $\widehat{\nabla}_{\mu}^{n+1}\phi$ constitutes an unbiased estimator of $\widehat{\nabla}_{\mu}\phi$, that is, a valid stochastic gradient associated with the latter. Using this fact, we now characterize the evolution of $\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2$, where $\boldsymbol{x}^o \in \mathcal{X}$ is an optimal solution of problem (51), provided such solution exists.

Lemma 9. (Third Zeroth-order SA Level: Optimality Error Growth) *Suppose that Assumption 2 is in effect, and let $\beta_n \in (0,1]$, $\gamma_n \in (0,1]$, for all $n \in \mathbb{N}$. Also, suppose that $\mathcal{X}_{\mu}^o \equiv \arg\min_{\boldsymbol{x}\in\mathcal{X}}\phi_{\mu}(\boldsymbol{x}) \neq \emptyset$ and consider any $\boldsymbol{x}^o \in \mathcal{X}_{\mu}^o$. Then, for every $0 < \mu \leq \mu_o$, there exists another problem dependent constant $\Sigma_p^4 < \infty$, also increasing and bounded in μ, such that the process $\{(\boldsymbol{x}^n, y^n, z^n)\}_{n\in\mathbb{N}}$ generated by the Free-MESSAGE^p algorithm satisfies*

$$\begin{aligned}&\mathbb{E}_{\mathscr{D}^n}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\}\\ &\quad\leq \big(1 + \Sigma_p^4 c^2\big(\alpha_n^2\beta_n^{-1} + \alpha_n^2\gamma_n^{-1}\mathbb{1}_{\{p>1\}}\big)\big)\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2 + \Sigma_p^1\alpha_n^2\\ &\qquad - 2\alpha_n\big(\phi(\boldsymbol{x}^n) - \phi_{\mu}^o\big) + \beta_n|y^n - s_{\mu}(\boldsymbol{x}^n)|^2 + \gamma_n|z^n - g_{\mu}(\boldsymbol{x}^n, y^n)|^2\mathbb{1}_{\{p>1\}},\end{aligned} \tag{64}$$

for all $n \in \mathbb{N}$, almost everywhere relative to $\mathcal{P}$, where $\phi_{\mu}^o \equiv \inf_{\boldsymbol{x}\in\mathcal{X}}\phi_{\mu}(\boldsymbol{x})$.

Proof of Lemma 9. See Appendix E.4. ∎

At this point, it is important to observe that Lemmata 6, 7, 8 and 9 share exactly the same structure with the corresponding results used in the analysis of the original *MESSAGE*^{p} algorithm of [22]; see, in particular, ([22], Section 4.4). Therefore, the behavior of the *Free-MESSAGE*^{p} algorithm as a method to solve *the surrogate problem* (51) can be analyzed *almost automatically*, by calling the respective convergence results developed in [22], which are based exclusively on the counterparts of Lemmata 6, 7, 8 and 9, presented therein. Then, the obtained results can be nicely related to the base problem (1), via Lemma 3. This is the path taken for proving our main results, as discussed below.

Also note that all constants Σ_p^1, Σ_p^2, Σ_p^3 and Σ_p^4 involved in Lemmata 6, 7, 8 and 9, respectively, *are all increasing and bounded in the smoothing parameter* $\mu \in (0, \mu_o]$. Therefore, when deriving convergence rates of the expected value type, based exclusively on Lemmata 6, 7, 8 and 9, similarly to ([22], Section 4.4, and Lemmata 3, 5, 6 and 7) and, as we will see, under appropriate stepsize initialization, all resulting constants will also be increasing and bounded functions of $\mu \in (0, \mu_o]$.

6.3 Path Convergence

The path behavior of the *Free-MESSAGE*^{p} algorithm may be characterized via the following result. Hereafter, let $\phi^* \triangleq \inf_{\boldsymbol{x}\in\mathcal{X}}\phi(\boldsymbol{x}) \in \mathbb{R}$.

Theorem 1. (Path Convergence of the *Free-MESSAGE*^{p} Algorithm) *Suppose that Assumption 2 is in effect, and let $\beta_n \in (0,1]$, $\gamma_n \in (0,1]$, for all $n \in \mathbb{N}$. Also, suppose that*

$$\sum_{n\in\mathbb{N}}\alpha_n \equiv \infty,\ \sum_{n\in\mathbb{N}}\alpha_n^2 + \beta_n^2 + \frac{\alpha_n^2}{\beta_n} < \infty, \text{ and if } p > 1,\ \sum_{n\in\mathbb{N}}\gamma_n^2 + \frac{\alpha_n^2}{\gamma_n} + \frac{\beta_n^2}{\gamma_n} < \infty. \tag{65}$$

Then, for $0 < \mu \leq \mu_o$*, and if* $\mathcal{X}_\mu^o \neq \emptyset$*, there exists an event* $\Omega' \subseteq \Omega$ *with* $\mathcal{P}(\Omega') \equiv 1$*, such that, for every* $\omega \in \Omega'$*, the process* $\{\boldsymbol{x}^n(\omega)\}_{n\in\mathbb{N}}$ *generated by the Free-MESSAGE*^{p} *algorithm converges as*

$$\boldsymbol{x}^n(\omega) \underset{n\to\infty}{\longrightarrow} \boldsymbol{x}^o(\omega) \in \mathcal{X}_\mu^o, \tag{66}$$

also implying that

$$\lim_{n\to\infty} \phi(\boldsymbol{x}^n(\omega)) - \phi^* \leq 2 \sup_{\boldsymbol{x}\in\mathcal{X}} |\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})| \equiv \Sigma^o \mu(\mu^\varepsilon + c). \tag{67}$$

In other words, almost everywhere relative to $\mathcal{P}$*,* $\{\boldsymbol{x}^n\}_{n\in\mathbb{N}}$ *converges to a point in the set of optimal solutions of (51), and* $\{\phi(\boldsymbol{x}^n)\}_{n\in\mathbb{N}}$ *converges to a linearly shrinking* μ*-neighborhood of* ϕ^**.*

Proof of Theorem 1. The proof of (66) follows directly from ([22], Section 4.4, Theorem 3), based on an application of the *T-level almost-supermartingale convergence lemma* [45]. To prove (67), note that, for every $\omega \in \Omega'$, continuity of ϕ on $\mathcal{X}$ implies that

$$\lim_{n\to\infty} \phi(\boldsymbol{x}^n(\omega)) - \phi^* \equiv \phi(\boldsymbol{x}^o(\omega)) - \phi^*. \tag{68}$$

Then, since $\boldsymbol{x}^o(\omega) \in \mathcal{X}_\mu^o$, Lemma 3 implies that

$$\begin{aligned}
\phi(\boldsymbol{x}^o(\omega)) - \phi^* &\equiv \phi(\boldsymbol{x}^o(\omega)) - \inf_{\boldsymbol{x}\in\mathcal{X}} \phi(\boldsymbol{x}) \\
&\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} \phi(\boldsymbol{x}^o(\omega)) - \phi(\boldsymbol{x}) \qquad (69)\\
&\leq \phi_\mu(\boldsymbol{x}^o(\omega)) - \phi_\mu(\boldsymbol{x}^o(\omega)) + \Sigma^o\mu(\mu^\varepsilon + c) \\
&\equiv \Sigma^o\mu(\mu^\varepsilon + c), \qquad (70)
\end{aligned}$$

and we are done. ■

6.4 Convergence Rates

6.4.1 Convex Cost

For the general case of a convex cost $F(\cdot, \boldsymbol{W})$, we have the following result on the rate of convergence of the *Free-MESSAGE*^{p} algorithm, concerning smoothened iterates of the form [44, 45]

$$\widehat{\boldsymbol{x}}^n \triangleq \frac{1}{\lceil n/2 \rceil} \sum_{i\in\mathbb{N}_n^{n-\lceil n/2\rceil}} \boldsymbol{x}^i, \quad n \in \mathbb{N}^+. \tag{71}$$

Theorem 2. (Rate | Convex Cost | Subharmonic Stepsizes) *Let Assumption 2 be in effect, set* $\alpha_0 \equiv \beta_0 \equiv \gamma_0 \equiv 1$*, and for* $n \in \mathbb{N}^+$*, choose* $\alpha_n \equiv n^{-\tau_1}$*,* $\beta_n \equiv n^{-\tau_2}$ *and* $\gamma_n \equiv n^{-\tau_3}$*, where, for fixed* $\epsilon \in [0, 1)$*,* $\delta \in (0, 1)$ *and* $\zeta \in (0, 1)$ *such that* $\delta \geq \zeta$*,*

$$\begin{cases} \tau_1 \equiv (3+\epsilon)/4 \quad \text{and} \quad \tau_2 \equiv (1+\delta\epsilon)/2, & \text{if } p \equiv 1 \\ \tau_1 \equiv (7+\epsilon)/8, \quad \tau_2 \equiv (3+\delta\epsilon)/4 \quad \text{and} \quad \tau_3 \equiv (1+\zeta\epsilon)/2, & \text{if } p > 1 \end{cases}. \tag{72}$$

Additionally, for $0 < \mu \leq \mu_o$*, suppose that* $\sup_{n\in\mathbb{N}} \mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\} \leq E_\mu < \infty$*,* $\boldsymbol{x}^o \in \mathcal{X}_\mu^o$*. Then, for every* $n \in \mathbb{N}^+$*, the Free-MESSAGE*^{p} *algorithm satisfies*

$$\mathbb{E}\{\phi(\widehat{\boldsymbol{x}}^n) - \phi^*\} \leq \mathcal{K}_p^{E_\mu} n^{-(1-\epsilon)/(4\mathbb{1}_{\{p\in(1,2]\}}+4)} + \Sigma^o\mu(\mu^\varepsilon + c), \tag{73}$$

where $\mathcal{K}_p^{E_\mu} \in (0, \infty)$ *is increasing and bounded in* μ*, whenever* E_μ *is in fact independent of* μ*.*

Proof of Theorem 2. As in the proof of Theorem 1, the result follows in part from ([22], Section 4.4, Theorem 4 and its proof), which applied to our setting yields

$$\mathbb{E}\{\phi_\mu(\widehat{\boldsymbol{x}}^n) - \phi_\mu^o\} \leq \mathcal{K}_p^{E_\mu} n^{-(1-\epsilon)/(4\mathbb{1}_{\{p\in(1,2]\}}+4)}, \quad \forall n \in \mathbb{N}^+, \tag{74}$$

where $\mathcal{K}_p^{E_\mu} \in (0,\infty)$ is increasing and bounded in μ, whenever E_μ is in fact not dependent on μ (for instance, whenever $\mathcal{X}$ is compact). Then, for any choice of $\boldsymbol{x}^o \in \mathcal{X}_\mu^o$, Lemma 3 implies that

$$\begin{aligned}\phi(\widehat{\boldsymbol{x}}^n) - \phi^* &\equiv \phi(\widehat{\boldsymbol{x}}^n) - \inf_{\boldsymbol{x}\in\mathcal{X}} \phi(\boldsymbol{x}) \\ &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} \phi(\boldsymbol{x}^o(\omega)) - \phi(\boldsymbol{x}) \\ &\leq \phi_\mu(\widehat{\boldsymbol{x}}^n) - \phi_\mu(\boldsymbol{x}^o) + \Sigma^o\mu(\mu^\varepsilon + c) \\ &\equiv \phi_\mu(\widehat{\boldsymbol{x}}^n) - \phi_\mu^o + \Sigma^o\mu(\mu^\varepsilon + c), \quad \forall n \in \mathbb{N}^+,\end{aligned} \tag{75}$$

everywhere on Ω. Taking expectations completes the proof. ■

6.4.2 Strongly Convex Cost

Next, we assume that $F(\cdot, \boldsymbol{W})$ is *σ-strongly convex* on $\mathbb{R}^N$. If subharmonic stepsizes are used, we have the next result, significantly improving Theorem 2. Hereafter, let $\boldsymbol{x}^* \triangleq \arg\min_{\boldsymbol{x}\in\mathcal{X}}\phi(\boldsymbol{x})$.

Theorem 3. (Rate | Strongly Convex Cost | Subharmonic Stepsizes) *Let Assumption 2 be in effect, and suppose that $F(\cdot, \boldsymbol{W})$ is σ-strongly convex on $\mathbb{R}^N$. Set $\alpha_0 \equiv \sigma^{-1}$ and $\beta_0 \equiv \gamma_0 \equiv 1$, and for $n \in \mathbb{N}^+$, choose $\alpha_n \equiv (\sigma n)^{-1}$, $\beta_n \equiv n^{-\tau_2}$ and $\gamma_n \equiv n^{-\tau_3}$, where, if $p \equiv 1$, $\tau_2 \equiv 2/3$, whereas if $p > 1$, and for fixed $\epsilon \in [0,1)$, and $\delta \in (0,1)$,*

$$\tau_2 \equiv (3+\epsilon)/4 \quad \textit{and} \quad \tau_3 \equiv (1+\delta\epsilon)/2. \tag{76}$$

Also define the quantity $n_o(\tau_2) \triangleq \lceil (1-\tau_2^{1/(\tau_2+1)})^{-1} \rceil \in \mathbb{N}^3$. Then, for $0 < \mu \leq \mu_o$ and for every $n \in \mathbb{N}^{n_o(\tau_2)}$, the Free-MESSAGE^p algorithm satisfies

$$\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^*\|_2^2\} \leq \Sigma_p^\sigma \times \begin{Bmatrix} (n_o(\tau_2)+3)n^{-2/3}, & \textit{if } p \equiv 1 \\ (n_o(\tau_2)+2(1-\epsilon)^{-1})n^{-(1-\epsilon)/2}, & \textit{if } p \in (1,2] \end{Bmatrix} + \frac{2\Sigma^o\mu(\mu^\varepsilon + c)}{\sigma}, \tag{77}$$

where $\Sigma_p^\sigma \in (0,\infty)$ is increasing and bounded in μ, and if $\sigma \geq 1$, $\Sigma_p^\sigma \leq \Sigma_p/\sigma^2 < \infty$.

Proof of Theorem 3. We focus on the case where $p \in (1,2]$; when $p \equiv 1$, the steps to the proof of the theorem are similar. First, we discuss the implications of assuming σ-strong convexity of $F(\cdot, \boldsymbol{W})$ on $\mathbb{R}^N$, which is equivalent to the condition

$$F(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}, \boldsymbol{w}) \leq \alpha F(\boldsymbol{x}, \boldsymbol{w}) + (1-\alpha)F(\boldsymbol{y}, \boldsymbol{w}) - \alpha(1-\alpha)\sigma\|\boldsymbol{x} - \boldsymbol{y}\|_2^2, \tag{78}$$

being true for all $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M$ and for all $\alpha \in [0,1]$. Indeed, for $F((\cdot) + \mu\boldsymbol{U}, \boldsymbol{W})$ we have

$$\begin{aligned}F(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} + \boldsymbol{u}, \boldsymbol{w}) &\equiv F(\alpha(\boldsymbol{x} + \boldsymbol{u}) + (1-\alpha)(\boldsymbol{y} + \boldsymbol{u}), \boldsymbol{w}) \\ &\leq \alpha F(\boldsymbol{x} + \boldsymbol{u}, \boldsymbol{w}) + (1-\alpha)F((\boldsymbol{y} + \boldsymbol{u}), \boldsymbol{w})\end{aligned}$$

$$
\begin{aligned}
&- \alpha(1-\alpha)\sigma\|(\boldsymbol{x}+\boldsymbol{u})-(\boldsymbol{y}+\boldsymbol{u})\|_2^2 \\
&\equiv \alpha F(\boldsymbol{x}+\boldsymbol{u},\boldsymbol{w}) + (1-\alpha)F(\boldsymbol{y}+\boldsymbol{u},\boldsymbol{w}) \\
&- \alpha(1-\alpha)\sigma\|\boldsymbol{x}-\boldsymbol{y}\|_2^2,
\end{aligned}
\tag{79}
$$

for all $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{w}) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M$ and for all $\alpha \in [0,1]$. This demonstrates that, for every $\mu \geq 0$, $F((\cdot)+\mu\boldsymbol{U},\boldsymbol{W})$ and thus $F((\cdot)+\mu\boldsymbol{U},\boldsymbol{W}) - \mu U$ are both strongly convex on $\mathbb{R}^N$ with the same parameter σ, independent of μ. Therefore, ([22], Proposition 5) implies that ϕ_μ is σ-strongly convex on $\mathbb{R}^N$, which is equivalent to the alternative condition

$$
\phi_\mu(\boldsymbol{x}) \geq \phi_\mu(\boldsymbol{y}) + (\nabla\phi_\mu(\boldsymbol{y}))^{\boldsymbol{T}}(\boldsymbol{x}-\boldsymbol{y}) + \sigma\|\boldsymbol{x}-\boldsymbol{y}\|_2^2,
\tag{80}
$$

being true for all $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N$. Additionally, since ϕ_μ is σ-strongly convex on $\mathbb{R}^N$, its infimum over $\mathcal{X}$ is attained for some unique $\boldsymbol{x}^o \in \mathcal{X}$ (depending on μ). As a result, it is true that

$$
\phi_\mu(\boldsymbol{x}) \geq \phi_\mu(\boldsymbol{x}^o) + (\nabla\phi_\mu(\boldsymbol{x}^o))^{\boldsymbol{T}}(\boldsymbol{x}-\boldsymbol{x}^o) + \sigma\|\boldsymbol{x}-\boldsymbol{x}^o\|_2^2, \quad \forall \boldsymbol{x} \in \mathcal{X}.
\tag{81}
$$

But by ([35], Theorem 3.33), and with the multifunction $\mathscr{N}_{\mathcal{X}} : \mathcal{X} \rightrightarrows \mathbb{R}^N$ being the normal cone to $\mathcal{X}$ defined as

$$
\mathscr{N}_{\mathcal{X}}(\overline{\boldsymbol{x}}) \triangleq \left\{\boldsymbol{z} \in \mathbb{R}^N \,\middle|\, \boldsymbol{z}^{\boldsymbol{T}}(\boldsymbol{x}-\overline{\boldsymbol{x}}) \leq 0,\ \forall \boldsymbol{x} \in \mathcal{X}\right\},
\tag{82}
$$

it follows that

$$
-\nabla\phi_\mu(\boldsymbol{x}^o) \in \mathscr{N}_{\mathcal{X}}(\boldsymbol{x}^o) \iff -\nabla\phi_\mu(\boldsymbol{x}^o)(\boldsymbol{x}-\boldsymbol{x}^o) \leq 0,\ \forall \boldsymbol{x} \in \mathcal{X}.
\tag{83}
$$

This last fact also implies that

$$
\phi_\mu(\boldsymbol{x}) - \phi_\mu^o \geq \sigma\|\boldsymbol{x}-\boldsymbol{x}^o\|_2^2, \quad \forall \boldsymbol{x} \in \mathcal{X}.
\tag{84}
$$

Similarly,

$$
\phi(\boldsymbol{x}) - \phi^* \geq \sigma\|\boldsymbol{x}-\boldsymbol{x}^*\|_2^2, \quad \forall \boldsymbol{x} \in \mathcal{X}.
\tag{85}
$$

Next, observe that, by our assumptions (in particular, Condition **C5**), in addition to the constants Σ_p^2, Σ_p^3 and Σ_p^4 involved in Lemmata 7, 8 and 9 being bounded and increasing in $\mu \in (0,\mu_o]$, the average errors $\mathbb{E}\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\}$ and $\mathbb{E}\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\}$ are *both* uniformly bounded relative to $n \in \mathbb{N}$ *and* $\sigma > 0$ *and* $\mu \in (0,\mu_o]$, and increasing relative to the latter, as well. Additionally, let us show that $\mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\}$ is also uniformly bounded relative to $n \in \mathbb{N}^+$ and increasing and bounded in $\mu \in (0,\mu_o]$, given our particular choice of $\alpha_0 \equiv \sigma^{-1}$. First, we exploit (84), and taking expectations on both sides of (199) (see proof of Lemma 9), we get

$$
\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\} \leq (1-2\sigma\alpha_n)\,\mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\} + \Sigma_p^1\alpha_n^2 + \mathbb{E}\{\boldsymbol{U}^{n+1}\},
\tag{86}
$$

being true for all $n \in \mathbb{N}$. Second, by (207) (once more, from the proof of Lemma 9), it is true that

$$
\mathbb{E}_{\mathscr{D}^n}\{\boldsymbol{U}^{n+1}\} \leq \sigma\alpha_n\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2 + \alpha_n\frac{\Sigma_p^4 c^2}{\sigma}\left(|y^n - s_\mu(\boldsymbol{x}^n)|^2 + |z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\right),
\tag{87}
$$

almost everywhere relative to $\mathcal{P}$. Again, taking expectations on both sides, we obviously have

$$
\mathbb{E}\{\boldsymbol{U}^{n+1}\} \leq \sigma\alpha_n\mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\} + \alpha_n\frac{\Sigma_p^4 c^2}{\sigma}\left(\mathbb{E}\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\} + \mathbb{E}\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\}\right),
\tag{88}
$$

for all $n \in \mathbb{N}$. Consequently, there is another constant $\Sigma_p^5 < \infty$, increasing and bounded in μ and independent of σ, such that

$$\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\} \le (1 - \sigma\alpha_n)\,\mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\} + \Sigma_p^1\alpha_n^2 + \Sigma_p^5 c^2 \frac{\alpha_n}{\sigma}, \tag{89}$$

for all $n \in \mathbb{N}$. By using the same inductive argument as in ([22], Section 4.4, last part of proof of Lemma 9), and by noting that

$$\begin{aligned}\mathbb{E}\{\|\boldsymbol{x}^1 - \boldsymbol{x}^o\|_2^2\} &\le (1 - \sigma\alpha_0)\,\mathbb{E}\{\|\boldsymbol{x}^0 - \boldsymbol{x}^o\|_2^2\} + \Sigma_p^1\alpha_0^2 + \Sigma_p^5 c^2 \frac{\alpha_0}{\sigma} \\ &\equiv \Sigma_p^1\sigma^{-2} + \Sigma_p^5 c^2\sigma^{-2},\end{aligned} \tag{90}$$

where the right-hand side is increasing and bounded in μ, it easily follows that

$$\sup_{n\in\mathbb{N}^+} \mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\} \le \Sigma_p^1\sigma^{-2} + \Sigma_p^5 c^2\sigma^{-2}. \tag{91}$$

This is all the "extras" we need. Now, by another closer inspection of ([22], Section 4.4, Lemma 9, Theorem 5 and the respective proofs), it follows that for $\mu \in (0, \mu_o]$ and for every $n \in \mathbb{N}^{n_o(\tau_2)} \subseteq \mathbb{N}^3$,

$$\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\} \le \overline{\Sigma}_p^\sigma(n_o(\tau_2) + 2\,(1-\epsilon)^{-1})n^{-(1-\epsilon)/2}, \tag{92}$$

for a problem dependent constant $\overline{\Sigma}_p^\sigma < \infty$, which, in case $\sigma \ge 1$, may be bounded as $\overline{\Sigma}_p^\sigma \le \overline{\Sigma}_p/\sigma^2$, for some other constant $\overline{\Sigma}_p$ (independent of σ). The constant $\overline{\Sigma}_p^\sigma$ is also increasing and bounded in μ, since it is dependent only on Σ_p^1, Σ_p^2, Σ_p^3 and Σ_p^4, as well as the uniform bounds of $\mathbb{E}\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\}$, $\mathbb{E}\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\}$, and $\mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\}$, as discussed above. Finally, we may exploit Lemma 3, and (85), to obtain

$$\begin{aligned}\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^*\|_2^2\} &\le 2\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\} + 2\|\boldsymbol{x}^o - \boldsymbol{x}^*\|_2^2 \\ &\le 2\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^o\|_2^2\} + 2\frac{1}{\sigma}(\phi\,(\boldsymbol{x}^o) - \phi^*) \\ &\le \Sigma_p^\sigma(n_o(\tau_2) + 2\,(1-\epsilon)^{-1})n^{-(1-\epsilon)/2} + \frac{2\Sigma^o\mu(\mu^\varepsilon + c)}{\sigma},\end{aligned} \tag{93}$$

being true for all $n \in \mathbb{N}^{n_o(\tau_2)}$, where $\Sigma_p^\sigma \triangleq 2\overline{\Sigma}_p^\sigma$. ■

We also provide a rate result for the case of constant stepsizes, very popular in practical considerations. This is useful in particular when the distribution of $\boldsymbol{W}$ changes during the operation of the algorithm, and the goal is to make the *Free-MESSAGE*^{p} algorithm *adaptive* to such changes. Also, let $\boldsymbol{x}^o \triangleq \arg\min_{\boldsymbol{x}\in\mathcal{X}} \phi_\mu\,(\boldsymbol{x})$.

Theorem 4. (Rate | Strongly Convex Cost | Constant Stepsizes) *Let Assumption 2 be in effect, and suppose that $F\,(\cdot, \boldsymbol{W})$ is σ-strongly convex on $\mathbb{R}^N$. For $n \in \mathbb{N}^+$, choose the stepsizes as $\alpha_n \equiv \alpha\sigma^{-1}, \alpha \in (0,1)$, $\beta_n \equiv \beta \in (0,1]$ and $\gamma_n \equiv \gamma \in (0,1]$, such that $\alpha < \min\{\beta, \gamma\}$. Then, for $0 < \mu \le \mu_o$ and for every $n \in \mathbb{N}^+$, the Free-MESSAGE^p algorithm satisfies*

$$\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^*\|_2^2\} \le (1-\alpha)^n\left(2\|\boldsymbol{x}^0 - \boldsymbol{x}^o\|_2^2 + \frac{\widehat{\Sigma}_p^1}{\sigma^2}\right) + \widehat{\Sigma}_p^\sigma\mathcal{H}(\alpha,\beta,\gamma) + \frac{2\Sigma^o\mu(\mu^\varepsilon + c)}{\sigma}, \tag{94}$$

where $\widehat{\Sigma}_p^1 \in (0,\infty)$ is independent of σ, $\widehat{\Sigma}_p^\sigma \in (0,\infty)$ is such that if $\sigma \ge 1$, $\widehat{\Sigma}_p^\sigma \le \widehat{\Sigma}_p^0/\sigma^2 < \infty$, both are increasing and bounded in μ, and $\mathcal{H}(\alpha,\beta,\gamma) \triangleq \alpha + \beta + \alpha^2\beta^{-2} + (\gamma + \alpha^2\gamma^{-2} + \beta^2\gamma^{-2})\mathbb{1}_{\{p\in(1,2]\}}$.

Proof of Theorem 4. Once more, we explicitly present the proof whenever $p \in (1,2]$. Let $J_s^n \triangleq \mathbb{E}\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\}$, $J_g^n \triangleq \mathbb{E}\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\}$, and $J_o^n \triangleq \mathbb{E}\{\|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2\}$, $n \in \mathbb{N}$, and for nonnegative sequences $\{H_s^n\}_{n\in\mathbb{N}}$ and $\{H_g^n\}_{n\in\mathbb{N}}$, define

$$J^n \triangleq J_o^n + H_s^{n-1} J_s^{n-1} + H_g^{n-1} J_g^{n-1}, \quad n \in \mathbb{N}^+. \tag{95}$$

Then, by our assumptions, and from ([22], Section 4.4, Lemma 9), it follows that $\{H_s^n\}_{n\in\mathbb{N}}$ and $\{H_g^n\}_{n\in\mathbb{N}}$ may be chosen in a way such that, for every $n \in \mathbb{N}^+$,

$$J^{n+1} \leq (1-\alpha) J^n + \widetilde{\Sigma}_p^\sigma \left(\alpha^2 + \frac{\alpha^3}{\beta^2} + \alpha\beta + \frac{\alpha^3}{\gamma^2} + \frac{\alpha\beta^2}{\gamma^2} + \alpha\gamma\right). \tag{96}$$

where $0 < \widetilde{\Sigma}_p^\sigma < \infty$ is increasing and bounded in μ. Proceeding inductively, we have

$$\begin{aligned}
J^{n+1} &\leq (1-\alpha) J^n + \widetilde{\Sigma}_p^\sigma \left(\alpha^2 + \frac{\alpha^3}{\beta^2} + \alpha\beta + \frac{\alpha^3}{\gamma^2} + \frac{\alpha\beta^2}{\gamma^2} + \alpha\gamma\right) \\
&\leq (1-\alpha)^2 J^{n-1} + \widetilde{\Sigma}_p^\sigma \left(\alpha^2 + \frac{\alpha^3}{\beta^2} + \alpha\beta + \frac{\alpha^3}{\gamma^2} + \frac{\alpha\beta^2}{\gamma^2} + \alpha\gamma\right)(1 + (1-\alpha)) \\
&\vdots \\
&\leq (1-\alpha)^n J^1 + \widetilde{\Sigma}_p^\sigma \left(\alpha^2 + \frac{\alpha^3}{\beta^2} + \alpha\beta + \frac{\alpha^3}{\gamma^2} + \frac{\alpha\beta^2}{\gamma^2} + \alpha\gamma\right) \sum_{i\in\mathbb{N}_{n-1}} (1-\alpha)^i \\
&\equiv (1-\alpha)^n J^1 + \widetilde{\Sigma}_p^\sigma \left(\alpha^2 + \frac{\alpha^3}{\beta^2} + \alpha\beta + \frac{\alpha^3}{\gamma^2} + \frac{\alpha\beta^2}{\gamma^2} + \alpha\gamma\right) \frac{1-(1-\alpha)^n}{\alpha} \\
&\leq (1-\alpha)^n J^1 + \widetilde{\Sigma}_p^\sigma \left(\alpha + \frac{\alpha^2}{\beta^2} + \beta + \frac{\alpha^2}{\gamma^2} + \frac{\beta^2}{\gamma^2} + \gamma\right).
\end{aligned} \tag{97}$$

Now, again from ([22], Section 4.4, Lemma 9 *and* its proof), and as in Theorem 3, it follows that, whenever $\sigma \geq 1$, $\widetilde{\Sigma}_p^\sigma \leq \widetilde{\Sigma}_p^0 / \sigma^2$, for some $\widetilde{\Sigma}_p^0 < \infty$, and the same type of argument holds for H_s^0 and H_g^0, as well, but for all $\sigma > 0$. Therefore, it is true that

$$\begin{aligned}
J_s^1 &\equiv J_o^1 + H_s^0 J_s^0 + H_g^0 J_g^0 \\
&\leq (1-\alpha) J_o^0 + \Sigma_p^1 \frac{\alpha^2}{\sigma^2} + c^2 \Sigma_p^5 \frac{\alpha}{\sigma^2} + H_s^0 J_s^0 + H_g^0 J_g^0 \\
&\leq J_o^0 + \frac{\widetilde{\Sigma}_p^1}{\sigma^2},
\end{aligned} \tag{98}$$

where $0 < \widetilde{\Sigma}_p^1 < \infty$ is independent of σ, and increasing and bounded in μ. As a result, we get

$$J_o^{n+1} \leq J^{n+1} \leq (1-\alpha)^n \left(J_o^0 + \frac{\widetilde{\Sigma}_p^1}{\sigma^2}\right) + \widetilde{\Sigma}_p^\sigma \left(\alpha + \beta + \gamma + \frac{\alpha^2}{\beta^2} + \frac{\alpha^2}{\gamma^2} + \frac{\beta^2}{\gamma^2}\right), \tag{99}$$

being true for all $n \in \mathbb{N}^+$. Finally, using the same argument as in (93), it follows that,

$$\mathbb{E}\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^*\|_2^2\} \leq (1-\alpha)^n \left(2\|\boldsymbol{x}^0 - \boldsymbol{x}^o\|_2^2 + \frac{\widehat{\Sigma}_p^1}{\sigma^2}\right)$$

$$+ \widehat{\Sigma}_p^{\sigma}\left(\alpha+\beta+\gamma+\frac{\alpha^2}{\beta^2}+\frac{\alpha^2}{\gamma^2}+\frac{\beta^2}{\gamma^2}\right)+\frac{2\Sigma^o\mu(\mu^{\varepsilon}+c)}{\sigma}, \tag{100}$$

for every $n \in \mathbb{N}^+$, where $\widehat{\Sigma}_p^1 \triangleq 2\widetilde{\Sigma}_p^1$, $\widehat{\Sigma}_p^{\sigma} \triangleq 2\widetilde{\Sigma}_p^{\sigma}$ and, whenever $\sigma \geq 1$, $\widehat{\Sigma}_p^{\sigma} \leq \widehat{\Sigma}_p^0/\sigma^2 \triangleq 2\widetilde{\Sigma}_p^0/\sigma^2$. The proof is now complete. ■

6.5 Discussion

First, let us comment on the role of $\epsilon \in [0,1)$ on the rates of Theorems 2 and 3, which, for $\epsilon \equiv 0$, are of the orders of $\mathcal{O}(n^{-1/(4\mathbb{1}_{\{p\in(1,2]\}}+4)}+\mu)$ (roughly) and $\mathcal{O}(n^{-1/2}+\mu)$, as $\mu \to 0$, respectively, the latter when $p \in (1,2]$. However, if $\epsilon \equiv 0$, the resulting stepsizes do not satisfy the conditions of Theorem 1, and path convergence of the *Free-MESSAGE^p* algorithm is not guaranteed (see also [22]). Nevertheless, if $\epsilon \in (0,1)$, rates *arbitrarily close* to the ones above can be achieved, while path convergence is simultaneously guaranteed, ensuring better algorithmic stability.

We would also like to emphasize the explicit dependence on σ on both terms appearing on the right of (77) and (94), implying that *strong convexity benefits both algorithmic and smoothing stability*. Of course, all rate bounds in (73), (77) and (94) present certain tradeoffs among μ, σ and N. In particular, the dependence on N appears of both terms on the right of (73), (77) and (94), and varies significantly relative to the associated $\left(\mathsf{D},\mathsf{T}\right)$-pair. This issue is discussed in detail in the next section.

7 Sample Complexity Bounds and Dependence on μ and N

In this section, we derive explicit sample complexity bounds for the *Free-MESSAGE^p* algorithm, which reveal its dependence on the decision dimension, N, which is *not* due to intrinsic problem structure, but due to the lack of gradient information. To do this, we restrict our attention to two common and very important cost function classes discussed in Appendix D, that is, the Lipschitz class and the smooth class.

Our results will be based on the detailed characterization of the quantities B_1 and B_2 of Lemma 5 relative to μ *and* N, which are the basis for defining the constants Σ_p^1, Σ_p^2, Σ_p^3 and Σ_p^4 involved in Lemmata 6, 7, 8 and 9, respectively. In Section 6.1 (and Appendix E) we have already discussed and used the fact that all aforementioned constants are increasing and bounded in μ, despite the lack of specific assumptions on the involved $\left(\mathsf{D},\mathsf{T}\right)$-pair. Now, by focusing either on Lipschitz or smooth functions, which are recovered by specific choices of associated $\left(\mathsf{D},\mathsf{T}\right)$-pairs, and by appropriately choosing μ, it will be possible to fully characterize the dependence of our convergence rates on N, effectively quantifying their effect on the behavior of the *Free-MESSAGE^p* algorithm.

For simplicity, in the following we assume a strongly convex cost, since this is a case of paramount importance in practice. However, similar results hold for the the general case of a convex cost. Also, in both of the subsections that follow, we prefer to develop our arguments in a discussion format, rather than presenting formal proofs to previously stated results. However, at the end each subsection, we will summarize our findings in the form of formal results.

7.1 Case Study #1: Lipschitz Class

As in Appendix D.1, we consider then class of functions satisfying the Lipschitz-like condition

$$\left\|F\left(\boldsymbol{x}_1,\boldsymbol{W}\right)-F\left(\boldsymbol{x}_2,\boldsymbol{W}\right)\right\|_{\mathcal{L}_2} \leq G\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|_2, \quad \forall(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^N\times\mathbb{R}^N, \tag{101}$$

with $\mathsf{D}(\cdot) \equiv \mathsf{d}(\cdot) \equiv \|\cdot\|_2$ and $\mathsf{T} \equiv \mathsf{t}_2 \equiv \mathsf{t}_4 \equiv 0$, that is, the associated (D,T)-pair is both uniformly 4-stable and 2-effective (actually stable), where the latter follows from the former. As described above, we start with the quantities B_1 and B_2 of Lemma 5. For B_1, we may write

$$\begin{aligned} B_1 &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\big\{\big(\mu^{\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\big)^2 \|\boldsymbol{U}\|_2^2\big\} \\ &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} G^2\mathbb{E}\big\{\|\boldsymbol{U}\|_2^4\big\} \\ &\equiv \mathcal{O}(N^2). \end{aligned} \tag{102}$$

For B_2, the situation is similar, but we take cases for $p\in[1,2]$. If $p\in(1,2]$, then

$$\begin{aligned} B_2 &\equiv p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\big)^{p+2}(\|\boldsymbol{U}\|_2^2+U^2)\big\} \\ &\quad+\big(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1\big)^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\big)^2(\|\boldsymbol{U}\|_2^2+U^2)\big\}\big) \\ &= p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(G\|\boldsymbol{U}\|_2+|U|\big)^{p+2}(\|\boldsymbol{U}\|_2^2+U^2)\big\} \\ &\quad+\big(\mathcal{R}(0)+2V_p+\mu G\mathbb{E}\{\|\boldsymbol{U}\|_2\}\big)^p\mathbb{E}\big\{\big(G\|\boldsymbol{U}\|_2+|U|\big)^2(\|\boldsymbol{U}\|_2^2+U^2)\big\}\big) \\ &\le \mu^p\mathcal{O}(N^{p/2+2})+\big(\mathcal{R}(0)+2V+\mu G\sqrt{N}\}\big)^p\mathcal{O}(N^2), \end{aligned} \tag{103}$$

Because it will be useful later on, we also define the quantity

$$\begin{aligned} B_2^h &\equiv p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\big)^{p+2}U^2\big\} \\ &\quad+\big(\mathcal{R}(0)+2V_p+\mu^{1+\varepsilon}G\mathcal{D}_1\big)^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})+|U|\big)^2U^2\big\}\big) \\ &= p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(G\|\boldsymbol{U}\|_2+|U|\big)^{p+2}U^2\big\} \\ &\quad+\big(\mathcal{R}(0)+2V_p+\mu G\mathbb{E}\{\|\boldsymbol{U}\|_2\}\big)^p\mathbb{E}\big\{\big(G\|\boldsymbol{U}\|_2+|U|\big)^2U^2\big\}\big) \\ &\le \mu^p\mathcal{O}(N^{p/2+1})+\big(\mathcal{R}(0)+2V+\mu G\sqrt{N}\}\big)^p\mathcal{O}(N) \end{aligned} \tag{104}$$

In case, however, $p\equiv 1$, it is true that

$$\begin{aligned} B_2 &\equiv \mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U})+\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})+|U|\big)^2(\|\boldsymbol{U}\|_2^2+U^2)\big\}\big) \\ &= \mathbb{E}\big\{\big(G\|\boldsymbol{U}\|_2+|U|\big)^2(\|\boldsymbol{U}\|_2^2+U^2)\big\}\big) \\ &\equiv \mathcal{O}(N^2), \end{aligned} \tag{105}$$

and similarly we may show that $B_2^h \equiv \mathcal{O}(N)$, as well.

Observe that the dependence of both B_2 and B_2^h on N is independent of μ, in the structurally simpler case where $p\equiv 1$. This fact gives us a rule for choosing μ when $p\in(1,2]$. Indeed, if we set $\mu\equiv\mathcal{M}/\sqrt{N}$, assumed hereafter, for some constant $\mathcal{M}>0$ to be fixed later, we obtain the bounds $B_2\equiv\mathcal{O}(N^2)$ and $B_2^h\equiv\mathcal{O}(N)$, which are uniform relative to p.

Let us see how B_1, B_2, and B_2^h shape the constants Σ_p^1, Σ_p^2, Σ_p^3 and Σ_p^4 of Lemmata 6, 7, 8 and 9. For Σ_p^1, we know from the proof of Lemma 6 that

$$\Sigma_p^1 \equiv \big(\sqrt{B_1}+c(\eta+\mathbb{1}_{\{p\equiv 1\}})^{1-p}\big(\sqrt{B_1B_2}+\sqrt{B_2}\big)\big)^2. \tag{106}$$

However, a closer look to the proof of Lemma 6 reveals that a better estimate of Σ_p^1, namely,

$$\Sigma_p^1 \equiv \Big(\sqrt{B_1}+c(\eta+\mathbb{1}_{\{p\equiv 1\}})^{1-p}\Big(\sqrt{B_1B_2^h}+\sqrt{B_2}\Big)\Big)^2$$

$$
\begin{aligned}
&\equiv \big(\mathcal{O}(N) + c(\eta + \mathbb{1}_{\{p\equiv 1\}})^{1-p}\big(\mathcal{O}(N^{3/2}) + \mathcal{O}(N)\big)\big)^2 \\
&\equiv \mathcal{O}(N^3). \qquad (107)
\end{aligned}
$$

For Σ_p^2, the proof of Lemma 7 implies that

$$
\begin{aligned}
\Sigma_p^2 &\equiv 2\max\{(V_1')^2, B_1\} \\
&\equiv 2\max\{(\mu^{1+\varepsilon} G\mathcal{D}_2 + \mathcal{T}_2 + V)^2, B_1\} \\
&\equiv 2\max\{(\mathcal{M}G + V)^2, B_1\} \\
&\equiv \mathcal{O}(N^2). \qquad (108)
\end{aligned}
$$

Let us proceed with Σ_p^3, which is relevant only when $p \in (1, 2]$. Indeed, from the proof of Lemma 8, in particular, by (190)-(195), it is easy to see that

$$
\begin{aligned}
\Sigma_p^3 &\equiv 2\max\Big\{\sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\{\big(\mathcal{R}(0) + 2V_p + \mu^{1+\varepsilon} G\mathcal{D}_1 + \mu^{1+\varepsilon} G\mathsf{d}(\boldsymbol{U}) + \mu\mathsf{t}_4(\boldsymbol{x}, \boldsymbol{U}) + \mu|u|\big)^{2p}\}, \\
&\qquad\qquad 2B_2\Sigma_p^1, 4B_2\big((V_1')^2 + (\mu^{1+\varepsilon} G\mathcal{D}_1 + V)^2\big)\Big\} \\
&\equiv 2\max\big\{\mathbb{E}\big\{\big(\mathcal{R}(0) + 2V_p + \mu G\mathbb{E}\{\|\boldsymbol{U}\|_2\} + \mu G\|\boldsymbol{U}\|_2 + \mu|u|\big)^{2p}\big\}, \\
&\qquad\qquad 2B_2\Sigma_p^1, 4B_2\big((\mu G\mathcal{D}_2 + \mathcal{T}_2 + V)^2 + (\mu G\mathcal{D}_1 + V)^2\big)\big\} \\
&\leq 2\max\big\{\mathbb{E}\big\{\big(\mathcal{R}(0) + 2V_p + \mu G\sqrt{N} + \mu G\|\boldsymbol{U}\|_2 + \mu|u|\big)^{2p}\big\}, \\
&\qquad\qquad 2B_2\Sigma_p^1, 4B_2\big((\mu G\sqrt{N} + V)^2 + (\mu G\sqrt{N} + V)^2\big)\big\} \\
&\leq \max\big\{\mathcal{O}(N^p), \mathcal{O}(N^5), \mathcal{O}(N^2)\big\} \\
&\equiv \mathcal{O}(N^5). \qquad (109)
\end{aligned}
$$

Lastly, from the proof of Lemma 9, it is true that

$$
\begin{aligned}
\Sigma_p^4 &\equiv \begin{cases} L^2(N + B_1), & \text{if } p \equiv 1 \\ (N + B_1)\big((1 + \sqrt{B_2})\eta^{(1-p)} \max\{L, p^{-1}(p-1)\eta^{-p}\sqrt{B_2}\}\big)^2, & \text{if } p \in (1, 2] \end{cases} \\
&\equiv \mathcal{O}(N^{2+4\mathbb{1}_{\{p\in(1,2]\}}}). \qquad (110)
\end{aligned}
$$

Next, we may observe that

$$
\begin{aligned}
\mathbb{E}\big\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\big\} &\leq \mathbb{E}\big\{(|y^n| + |s_\mu(\boldsymbol{x}^n)|)^2\big\} \\
&\leq 4(\mu^{1+\varepsilon} G\mathcal{D}_1 + V)^2 \\
&\leq 4(\mathcal{M}G + V)^2 \equiv \mathcal{O}(1), \qquad (111)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\big\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\big\} &\leq \mathbb{E}\big\{(|z^n| + |g_\mu(\boldsymbol{x}^n, y^n)|)^2\big\} \\
&\leq 4(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon} G(\mathcal{D}_1 + \mathcal{D}_2) + \mu(\mathcal{T}_2 + 1))^{2p} \\
&\leq 4(\mathcal{R}(0) + 2V + 2\mathcal{M}G + \mu)^{2p} \equiv \mathcal{O}(1). \qquad (112)
\end{aligned}
$$

Now, we have all the required information in order to calculate the dependence on N of the constant Σ_p^σ, present in Theorem 3. To do this, we perform a very careful backtracking procedure in the proof of Theorem 3 (see Section 6.4.2), bookkeeping the complexity of the constants involved in the respective recursions, *plus* the complexity of initial conditions. Then, by closely reexamining ([22], Section 4.4, Lemma 9, Theorem 5 *and* the respective proofs), we may deduce that

$$\Sigma_p^\sigma \equiv \mathcal{O}(N^{4+7\mathbb{1}_{\{p\in(1,2]\}}}). \tag{113}$$

Further, if $\sigma \geq 1$, it is also true that $\Sigma_p^\sigma \leq \Sigma_p/\sigma^2 \equiv \mathcal{O}(N^{4+7\mathbb{1}_{\{p\in(1,2]\}}})/\sigma^2$. The next result summarizes our discussion above, providing a complexity estimate of the *Free-MESSAGE*^{p} algorithm, explicitly showing the dependence on N, for the case of the Lipschitz class.

Theorem 5. (Rate | Lipschitz & Strongly Convex Cost | Subharmonic Stepsizes) *Let Assumption 2 be in effect, and suppose that $F(\cdot,\boldsymbol{W})$ is σ-strongly convex on $\mathbb{R}^N$. Set $\alpha_0 \equiv \sigma^{-1}$ and $\beta_0 \equiv \gamma_0 \equiv 1$, and for $n \in \mathbb{N}^+$, choose $\alpha_n \equiv (\sigma n)^{-1}$, $\beta_n \equiv n^{-\tau_2}$ and $\gamma_n \equiv n^{-\tau_3}$, where, if $p \equiv 1$, $\tau_2 \equiv 2/3$, whereas if $p > 1$, and for fixed $\epsilon \in [0,1)$, and $\delta \in (0,1)$,*

$$\tau_2 \equiv (3+\epsilon)/4 \quad \textit{and} \quad \tau_3 \equiv (1+\delta\epsilon)/2. \tag{114}$$

Also define the quantity $n_o(\tau_2) \triangleq \lceil (1-\tau_2^{1/(\tau_2+1)})^{-1} \rceil \in \mathbb{N}^3$. Pick any $\delta > 0$ and choose

$$\mu \equiv \frac{\mathcal{M}}{\sqrt{N}}, \textit{ with } \mathcal{M} > 0 \textit{ sufficiently small, such that } \Sigma_*^o(\mathcal{M})\mathcal{M} \leq \frac{\sigma\delta}{4(1+c)}, \tag{115}$$

where, for the particular choice of μ, $\Sigma_*^o(\mathcal{M}) \triangleq \sup_{N\geq 1} \Sigma^o/\sqrt{N}$. Then, as long as

$$n \geq \begin{cases} \max\left\{n_o(\tau_2), \left(\dfrac{2\Sigma_p^\sigma}{\delta}\left(n_o(\tau_2)+3\right)\right)^{3/2}\right\} \equiv \mathcal{O}\left(\left(\dfrac{N^4}{\delta}\right)^{3/2}\right), & \textit{if } p \equiv 1 \\ \max\left\{n_o(\tau_2), \left(\dfrac{2\Sigma_p^\sigma}{\delta}\left(n_o(\tau_2)+\dfrac{2}{1-\epsilon}\right)\right)^{2/(1-\epsilon)}\right\} \equiv \mathcal{O}\left(\left(\dfrac{N^{11}}{\delta}\right)^{2/(1-\epsilon)}\right), & \textit{if } p \in (1,2] \end{cases}, \tag{116}$$

the Free-MESSAGE^p algorithm satisfies $\mathbb{E}\{\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^*\|_2^2\} \leq \delta$.

In comparison with the risk-neutral setup, where $c \equiv 0$, the constant corresponding to Σ_p^σ is of the order of $\mathcal{O}(N^2)$, since only B_1 is involved in the respective calculations [28]. Also, the resulting rate climbs to the order of $\mathcal{O}(N^2)/n$, with a complexity estimate of the order of $\mathcal{O}(N^2/\delta)$ iterations.

We would like to emphasize that the increased orderwise dependence on N in Theorem 5 may be justified first by the increased difficulty of the risk-aware learning task, and second by the fact that nothing more than mere Lipschitz continuity has been imposed on the cost function $F(\cdot,\boldsymbol{W})$ (in addition to strong convexity). Most probably, the dependence on N can be improved by designing more sophisticated versions of *Free-MESSAGE*^{p}, possibly using ideas such as averaging, minibatching, or multi-point finite differences for gradient approximation.

The respective version of Theorem 4 (rate with constant stepsizes) may be formulated by following almost the same procedure as above, and is therefore omitted in our discussion.

7.2 Case Study #2: Smooth Class

We now consider the class consisting of functions obeying the smoothness-like condition

$$\left\|F\left(\boldsymbol{x}_1,\boldsymbol{W}\right)-F\left(\boldsymbol{x}_2,\boldsymbol{W}\right)-\left(\nabla F\left(\boldsymbol{x}_2,\boldsymbol{W}\right)\right)^{\boldsymbol{T}}\left(\boldsymbol{x}_1-\boldsymbol{x}_2\right)\right\|_{\mathcal{L}_2}\le G\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|_2^2, \tag{117}$$

for all $(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^N\times\mathbb{R}^N$. In Appendix D.2, we associated this class with the (D,T)-pair where $\mathsf{D}\left(\cdot\right)\equiv\left\|\cdot\right\|_2^2$ and $\mathsf{T}([\bullet,\star],\cdot)\equiv\left(\nabla F\left(\bullet,\star\right)\right)^{\boldsymbol{T}}(\cdot)$, and we showed that the associated (D,T)-pair is uniformly 2-effective on $\mathcal{X}$ whenever

$$\sup_{\boldsymbol{x}\in\mathcal{X}}\left\|\left\|\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right\|_2\right\|_{\mathcal{L}_2}<\infty, \tag{118}$$

and with the choices

$$\mathsf{d}\left(\boldsymbol{u}\right)\equiv\left\|\boldsymbol{u}\right\|_2^2\quad\text{and} \tag{119}$$

$$\mathsf{t}_2\left(\boldsymbol{x},\boldsymbol{u}\right)\equiv\left\|\left(\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right)^{\boldsymbol{T}}\boldsymbol{u}\right\|_{\mathcal{L}_2}\equiv\sqrt{\boldsymbol{u}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{u}}, \tag{120}$$

where

$$\nabla\mathsf{F}(\boldsymbol{x})\equiv\mathbb{E}\{\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\left(\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right)^{\boldsymbol{T}}\}, \tag{121}$$

and

$$\mathcal{T}_2\equiv\sup_{\boldsymbol{x}\in\mathcal{X}}\left\|\mathsf{t}_2\left(\boldsymbol{x},\boldsymbol{U}\right)\right\|_{\mathcal{L}_2}\equiv\sup_{\boldsymbol{x}\in\mathcal{X}}\left\|\left\|\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right\|_2\right\|_{\mathcal{L}_2}\equiv\mathcal{O}(1). \tag{122}$$

for all $(\boldsymbol{x},\boldsymbol{u})\in\mathcal{X}\times\mathbb{R}^N$. Let us show that, if (118) is true, the (D,T)-pair under consideration is uniformly 2-stable, as well. Indeed, one the one hand we have

$$\mathbb{E}\{(\mathsf{d}\left(\boldsymbol{U}\right))^2\left\|\boldsymbol{U}\right\|_2^2\}\equiv\mathbb{E}\{\left\|\boldsymbol{U}\right\|_2^6\}\equiv\mathcal{O}(N^3)\iff\mathsf{d}\left(\boldsymbol{U}\right)\left\|\boldsymbol{U}\right\|_2\in\mathcal{Z}_2, \tag{123}$$

while, on the other, it is true that

$$\begin{aligned}
\mathbb{E}\{(\mathsf{t}_2\left(\boldsymbol{x},\boldsymbol{U}\right))^2\left\|\boldsymbol{U}\right\|_2^2\}&\equiv\mathbb{E}\{\boldsymbol{U}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{U}\left\|\boldsymbol{U}\right\|_2^2\}\\
&\equiv\mathbb{E}\{\mathrm{tr}(\boldsymbol{U}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{U}\left\|\boldsymbol{U}\right\|_2^2)\}\\
&\equiv\mathbb{E}\{\mathrm{tr}(\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\left\|\boldsymbol{U}\right\|_2^2\nabla\mathsf{F}(\boldsymbol{x}))\}\\
&\equiv\mathrm{tr}(\mathbb{E}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\left\|\boldsymbol{U}\right\|_2^2\}\nabla\mathsf{F}(\boldsymbol{x}))\\
&=(N+2)\mathrm{tr}(\nabla\mathsf{F}(\boldsymbol{x})) \qquad (124)\\
&=(N+2)\left\|\left\|\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right\|_2\right\|_{\mathcal{L}_2}^2\\
&\equiv(N+2)\left\|\mathsf{t}_2\left(\boldsymbol{x},\boldsymbol{U}\right)\right\|_{\mathcal{L}_2}<\infty, \qquad (125)
\end{aligned}$$

which is equivalent to $\mathsf{t}_q\left(\cdot,\boldsymbol{U}\right)\left\|\boldsymbol{U}\right\|_2$ being *uniformly* in $\mathcal{Z}_2$, where (124) is due to (167) (Section C).

Further, we now show that the involved (D,T)-pair is uniformly 4-stable under the natural condition

$$\sup_{\boldsymbol{x}\in\mathcal{X}}\left\|\left\|\nabla F\left(\boldsymbol{x},\boldsymbol{W}\right)\right\|_2\right\|_{\mathcal{L}_4}<\infty, \tag{126}$$

which, of course, implies (118). Uniform 4-stability will be extremely crucial in our subsequent complexity analysis, as we will shortly see. The situation regarding d is as before, that is,

$$\mathbb{E}\{(\mathsf{d}\left(\boldsymbol{U}\right))^4\}\equiv\mathbb{E}\{\left\|\boldsymbol{U}\right\|_2^8\}\equiv\mathcal{O}(N^4)\iff\mathsf{d}\left(\boldsymbol{U}\right)\in\mathcal{Z}_4 \tag{127}$$

$$\mathbb{E}\{(\mathsf{d}(\boldsymbol{U}))^{\overline{q}}\|\boldsymbol{U}\|_2^2\} \equiv \mathbb{E}\{\|\boldsymbol{U}\|_2^{2\overline{q}+2}\} \equiv \mathcal{O}(N^{\overline{q}+1}) \iff \mathsf{d}(\boldsymbol{U})\|\boldsymbol{U}\|_2^{2/\overline{q}} \in \mathcal{Z}_{\overline{q}},\ \forall \overline{q} \in [2,4]. \tag{128}$$

The difficulty here comes from the function t_4, which is naturally defined as

$$\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u}) \equiv \|(\nabla F(\boldsymbol{x},\boldsymbol{W}))^{\boldsymbol{T}}\boldsymbol{u}\|_{\mathcal{L}_4}, \tag{129}$$

for all $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X} \times \mathbb{R}^N$. First, we are interested in an efficient estimate for $\|\mathsf{t}_4(\cdot,\boldsymbol{U})\|_{\mathcal{L}_4}$. For every $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X} \times \mathbb{R}^N$, we have

$$\begin{aligned}
(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u}))^4 &\equiv \mathbb{E}\{((\nabla F(\boldsymbol{x},\boldsymbol{W}))^{\boldsymbol{T}}\boldsymbol{u})^4\} \\
&\equiv \mathbb{E}\Big\{\Big(\sum_i \nabla_{x_i} F(\boldsymbol{x},\boldsymbol{W})\,u_i\Big)^4\Big\} \\
&\equiv \mathbb{E}\Big\{\sum_{i,j,k,l} \nabla_{x_i}^{\boldsymbol{W}}F\nabla_{x_j}^{\boldsymbol{W}}F\nabla_{x_k}^{\boldsymbol{W}}F\nabla_{x_l}^{\boldsymbol{W}}F u_i u_j u_k u_l\Big\} \\
&\equiv \sum_{i,j,k,l} \mathbb{E}\{\nabla_{x_i}^{\boldsymbol{W}}F\nabla_{x_j}^{\boldsymbol{W}}F\nabla_{x_k}^{\boldsymbol{W}}F\nabla_{x_l}^{\boldsymbol{W}}F\} u_i u_j u_k u_l \\
&\triangleq \sum_{i,j,k,l} \nabla_{i,j,k,l}^{\boldsymbol{x}} u_i u_j u_k u_l,
\end{aligned} \tag{130}$$

where, for brevity, we have defined $\nabla_{x_i}^{\boldsymbol{W}}F \triangleq \nabla_{x_i}F(\boldsymbol{x},\boldsymbol{W}), i \in \mathbb{N}_N^+$, and where all involved expectations are legitimate (for instance, by Cauchy-Schwarz). Therefore, we may integrate one more time and carefully write (again, all integrals exist)

$$\begin{aligned}
\mathbb{E}\{(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}))^4\} &\equiv \sum_{i,j,k,l} \nabla_{i,j,k,l}^{\boldsymbol{x}} \mathbb{E}\{U_i U_j U_k U_l\} \\
&< 9\sum_{i,j} \mathbb{E}\{(\nabla_{x_i}^{\boldsymbol{W}}F)^2(\nabla_{x_j}^{\boldsymbol{W}}F)^2\} \\
&\equiv 9\mathbb{E}\Big\{\sum_{i,j} (\nabla_{x_i}^{\boldsymbol{W}}F)^2(\nabla_{x_j}^{\boldsymbol{W}}F)^2\Big\} \\
&\equiv 9\mathbb{E}\Big\{\Big(\sum_i (\nabla_{x_i}^{\boldsymbol{W}}F)^2\Big)^2\Big\} \\
&\equiv 9\mathbb{E}\{\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2^4\},
\end{aligned} \tag{131}$$

for all $\boldsymbol{x} \in \mathcal{X}$. As a result, we get that

$$\begin{aligned}
\sup_{\boldsymbol{x}\in\mathcal{X}} \|\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\|_{\mathcal{L}_4} &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} (\mathbb{E}\{(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}))^4\})^{1/4} \\
&< 9^{1/4} \sup_{\boldsymbol{x}\in\mathcal{X}} (\mathbb{E}\{\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2^4\})^{1/4} \\
&\equiv 9^{1/4} \sup_{\boldsymbol{x}\in\mathcal{X}} \big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_4} \\
&< 2 \sup_{\boldsymbol{x}\in\mathcal{X}} \big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_4} < \infty.
\end{aligned} \tag{132}$$

What is more, the previous inequality shows that, in fact, $\sup_{\boldsymbol{x}\in\mathcal{X}} \|\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\|_{\mathcal{L}_4} \equiv \mathcal{O}(1)$ (independent of N). Second, using the fact that

$$\mathsf{t}_4(\boldsymbol{x},\boldsymbol{u}) \le \big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_4}\|\boldsymbol{u}\|_2, \tag{133}$$

then, for every $\overline{q} \in [2,4]$, it holds that

$$\|\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U})\,\|\boldsymbol{U}\|_2^{2/\overline{q}}\|_{\mathcal{L}_{\overline{q}}} \le \big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\,\|_2\big\|_{\mathcal{L}_4}\|\,\|\boldsymbol{U}\|_2^{2/\overline{q}+1}\|_{\mathcal{L}_{\overline{q}}}, \tag{134}$$

which implies that

$$\sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\{(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}))^{\overline{q}}\,\|\boldsymbol{U}\|_2^2\} \equiv \mathcal{O}(N^{\overline{q}/2+1}). \tag{135}$$

This is probably a suboptimal estimate, but it will serve our purposes well. Note, though, that when $\overline{q} \equiv 2$, (131) provides the improved estimate

$$\begin{aligned}\mathbb{E}\{(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}))^2\,\|\boldsymbol{U}\|_2^2\} &\le \sqrt{\mathbb{E}\{(\mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}))^4\}\mathbb{E}\{\|\boldsymbol{U}\|_2^4\}}\\ &\equiv \mathcal{O}(N).\end{aligned} \tag{136}$$

So far, we have shown that the (D,T)-pair associated to the smooth class (those functions satisfying (117)) is uniformly 4-stable, and we have provided appropriate estimates for the quantities involved.

Next, we proceed with our complexity estimates. As we did in the Lipschitz case above (Section 7.1), we start with the quantities B_1 and B_2 of Lemma 5. Note that, in what follows, we use the fact that the involved (D,T)-pair is both uniformly 2-stable *and* uniformly 4-stable, as previously discussed. For B_1, we have

$$\begin{aligned}B_1 &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\big)^2\,\|\boldsymbol{U}\|_2^2\big\}\\ &\equiv \sup_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\big)^2\,\|\boldsymbol{U}\|_2^2\big\}\\ &\equiv \mu^2\mathcal{O}(N^3) + \mathcal{O}(N).\end{aligned} \tag{137}$$

For B_2, and if $p \in (1,2]$, then

$$\begin{aligned}B_2 &\equiv p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^{p+2}(\|\boldsymbol{U}\|_2^2 + U^2)\big\}\\ &\quad + \big(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon}G\mathcal{D}_1\big)^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^2(\|\boldsymbol{U}\|_2^2 + U^2)\big\}\big)\\ &= p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^{p+2}(\|\boldsymbol{U}\|_2^2 + U^2)\big\}\\ &\quad + \big(\mathcal{R}(0) + 2V + \mu^2G\mathbb{E}\{\|\boldsymbol{U}\|_2^2\}\big)^p\mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^2(\|\boldsymbol{U}\|_2^2 + U^2)\big\}\big)\\ &\le \mu^{2p+2}\mathcal{O}(N^{p+3}) + \mu^p\mathcal{O}(N^{p/2+2}) + \big(\mathcal{R}(0) + 2V + \mu^2GN\big\}\big)^p(\mu^2\mathcal{O}(N^3) + \mathcal{O}(N)),\end{aligned} \tag{138}$$

As before, we also define the quantity

$$\begin{aligned}B_2^h &\equiv p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^{p+2}U^2\big\}\\ &\quad + \big(\mathcal{R}(0) + 2V + \mu^{1+\varepsilon}G\mathcal{D}_1\big)^p\mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^2U^2\big\}\big)\\ &= p^2\eta^{(p-2)}2^{p-1}\big(\mu^p\mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^{p+2}U^2\big\}\\ &\quad + \big(\mathcal{R}(0) + 2V + \mu^2G\mathbb{E}\{\|\boldsymbol{U}\|_2^2\}\big)^p\mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_4(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^2U^2\big\}\big)\\ &\le \mu^{2p+2}\mathcal{O}(N^{p+2}) + \mu^p\mathcal{O}(1) + \big(\mathcal{R}(0) + 2V + \mu^2GN\big\}\big)^p(\mu^2\mathcal{O}(N^2) + \mathcal{O}(1))\end{aligned} \tag{139}$$

When $p \equiv 1$, though, we get

$$B_2 \equiv \mathbb{E}\big\{\big(\mu^{\varepsilon}G\mathsf{d}(\boldsymbol{U}) + \mathsf{t}_2(\boldsymbol{x},\boldsymbol{U}) + |U|\big)^2(\|\boldsymbol{U}\|_2^2 + U^2)\big\}\big)$$

$$
\begin{aligned}
&= \mathbb{E}\big\{\big(\mu G\,\|\boldsymbol{U}\|_2^2 + \mathsf{t}_2\,(\boldsymbol{x},\boldsymbol{U}) + |U|\,\big)^2\big(\|\boldsymbol{U}\|_2^2 + U^2\big)\big\}\big) \\
&\equiv \mu^2\mathcal{O}(N^3) + \mathcal{O}(N),
\end{aligned} \tag{140}
$$

and similarly, $B_2^h \equiv \mu^2\mathcal{O}(N^2) + \mathcal{O}(1)$, as well.

Again, the case when $p \equiv 1$ provides guidance for choosing μ. In particular, if we set $\mu \equiv \mathcal{M}/N^{3/2}$, which is assumed hereafter, for some constant $\mathcal{M} > 0$, we obtain the bounds $B_1 \equiv \mathcal{O}(N)$, $B_2 \equiv \mathcal{O}(N)$ and $B_2^h \equiv \mathcal{O}(1)$. Again, as in Section 7.1, these bounds are uniform relative to p.

These improved complexity estimates on B_1, B_2, and B_2^h have substantial effects on constants Σ_p^1, Σ_p^2, Σ_p^3 and Σ_p^4 of Lemmata 6, 7, 8 and 9. To reveal those effects, we follow the same procedure as in Section 7.1. Specifically, it is true that

- $\Sigma_p^1 \equiv \mathcal{O}(N)$.
- $\Sigma_p^2 \equiv \mathcal{O}(N)$.
- $\Sigma_p^3 \equiv \mathcal{O}(N^2)$.
- $\Sigma_p^4 \equiv \mathcal{O}(N^{1+2\mathbb{1}_{\{p\in(1,2]\}}})$.
- $\mathbb{E}\big\{|y^n - s_\mu(\boldsymbol{x}^n)|^2\big\} \equiv \mathcal{O}(1)$ and $\mathbb{E}\big\{|z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2\big\} \equiv \mathcal{O}(1)$.

Then, we may again calculate the dependence of the constant Σ_p^σ on N, resulting in the estimate

$$
\Sigma_p^\sigma \equiv \mathcal{O}(N^{2+3\mathbb{1}_{\{p\in(1,2]\}}}). \tag{141}
$$

As before, if $\sigma \geq 1$, it is also true that $\Sigma_p^\sigma \leq \Sigma_p/\sigma^2 \equiv \mathcal{O}(N^{2+3\mathbb{1}_{\{p\in(1,2]\}}})/\sigma^2$, and we have the following result providing a complexity estimate of the *Free-MESSAGE*^{p} algorithm for the case of the smooth function class under study.

Theorem 6. (Rate | Smooth & Strongly Convex Cost | Subharmonic Stepsizes) *Let Assumption 2 be in effect, and suppose that $F(\cdot,\boldsymbol{W})$ is σ-strongly convex on $\mathbb{R}^N$. Set $\alpha_0 \equiv \sigma^{-1}$ and $\beta_0 \equiv \gamma_0 \equiv 1$, and for $n \in \mathbb{N}^+$, choose $\alpha_n \equiv (\sigma n)^{-1}$, $\beta_n \equiv n^{-\tau_2}$ and $\gamma_n \equiv n^{-\tau_3}$, where, if $p \equiv 1$, $\tau_2 \equiv 2/3$, whereas if $p > 1$, and for fixed $\epsilon \in [0,1)$, and $\delta \in (0,1)$,*

$$
\tau_2 \equiv (3+\epsilon)/4 \quad \text{and} \quad \tau_3 \equiv (1+\delta\epsilon)/2. \tag{142}
$$

Also define the quantity $n_o(\tau_2) \triangleq \lceil (1-\tau_2^{1/(\tau_2+1)})^{-1} \rceil \in \mathbb{N}^3$. Pick any $\delta > 0$ and choose

$$
\mu \equiv \frac{\mathcal{M}}{N^{3/2}}, \text{ with } \mathcal{M} > 0 \text{ sufficiently small, such that } \Sigma_*^o(\mathcal{M})\mathcal{M}\big(\mathcal{M}+c\big) \leq \frac{\sigma\delta}{4}. \tag{143}
$$

where, for the particular choice of μ, $\Sigma_*^o(\mathcal{M}) \triangleq \sup_{N\geq 1} \Sigma^o/N^{3/2}$. Then, as long as

$$
n \geq \begin{cases} \max\left\{n_o(\tau_2), \left(\dfrac{2\Sigma_p^\sigma}{\delta}\big(n_o(\tau_2)+3\big)\right)^{3/2}\right\} \equiv \mathcal{O}\left(\left(\dfrac{N^2}{\delta}\right)^{3/2}\right), & \text{if } p \equiv 1 \\ \max\left\{n_o(\tau_2), \left(\dfrac{2\Sigma_p^\sigma}{\delta}\left(n_o(\tau_2)+\dfrac{2}{1-\epsilon}\right)\right)^{2/(1-\epsilon)}\right\} \equiv \mathcal{O}\left(\left(\dfrac{N^5}{\delta}\right)^{2/(1-\epsilon)}\right), & \text{if } p \in (1,2] \end{cases}, \tag{144}
$$

the Free-MESSAGE^p algorithm satisfies $\mathbb{E}\big\{\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^*\|_2^2\big\} \leq \delta$.

Again, the respective version of Theorem 4 is omitted in our discussion, for brevity. In comparison with the risk-neutral setup, where $c \equiv 0$, the respective constant corresponding to Σ_p^σ is of the order of $\mathcal{O}(N)$. This gives a rate of the order of $\mathcal{O}(N)/n$, with a complexity estimate of the order of $\mathcal{O}(N/\delta)$ iterations. In comparison with the Lipschitz class, as studied in Section 7.1, we observe a very significant improvement in the case of smooth functions. Specifically, when $p \equiv 1$, smooth functions require an order of $\mathcal{O}(N^3)$ less iterations in order to reach the same expected solution accuracy, whereas, when $p \in (1, 2]$, this gap increases to an order of $\mathcal{O}(N^{12/(1-\epsilon)})$ (!). Therefore, smooth cost functions result in much more well-conditioned risk-aware problems of the form of (1), at least in regard to the efficiency of the *Free-MESSAGE^p* algorithm.

Of course, at this point there is no indication that any of our complexity estimates achieve some form of optimality; in fact, it is most probable that the dependence on N can be improved, even in the smooth case. However, this remains an open problem, subject to future investigation.

As a final comment, we would like to emphasize that, although the dependence on N is somewhat large in the results for both the Lipschitz and smooth function classes, in practice we expect a much better scalability of the *Free-MESSAGE^p* algorithm since, in most cases, the cost function and the risk regularizer would be chosen such that the conditioning of the problem is appropriate. Yet, our complexity bounds indeed reveal that risk-aware learning is fundamentally more complex than ordinary, risk-neutral leaning; in our work, this is clearly due to the compositional nature of the base problem (1).

8 Future Work

There are several interesting topics for future work, building on the results presented in this paper; indicatively, we discuss some. First, although our rate results quantify explicitly the dependence on μ and σ, we have not paid much attention to the decision dimension, N. Indeed, as we briefly discuss in Section 7, if $c \equiv 0$, then, orderwise relative to N, our bounds are equivalent to those in [28], known to be order-suboptimal (see, e.g., [11]). Therefore, it would be of interest to see if order improvement relative to N is possible, by potentially exploiting ideas from more ingenious methods for risk-neutral zeroth-order optimization, such as those with diminishing μ, multi-point finite differences, and/or minibatching. Second, also driven by [11], another challenging topic is the development of lower complexity bounds for risk-aware learning, which would be useful in the design of optimal algorithms and, of course, as complexity benchmarks. Lastly, relaxing the convexity of the base problem is of particular interest, as the resulting setting fits more accurately application settings in modern artificial intelligence and deep learning.

Appendix

A Two Toy-Examples of non-Lipschitz, non-Smooth, but SLipschitz Functions

Let $f(\cdot) \equiv (\cdot)^4$. Indeed, for every compact (say) set $\mathcal{F} \subset \mathbb{R}$, and for every $x \in \mathcal{F}$, we have

$$\begin{aligned}\left|f(x+u) - f(x) - 4x^3u - 4xu^3\right| &\equiv \left|(x+u)^4 - x^4 - 4x^3u - 4xu^3\right| \\ &\equiv \left|6x^2u^2 + u^4\right| \\ &\leq 6 \sup_{x \in \mathcal{F}} x^2u^2 + u^4\end{aligned}$$

$$\leq \max\left\{6 \sup_{x\in\mathcal{F}} x^2, 1\right\}\left(u^2+u^4\right)$$

$$\triangleq L_{\mathcal{F}}\big(u^2+u^4\big) \triangleq L_{\mathcal{F}}\mathsf{D}\left(u\right), \tag{145}$$

for all $u \in \mathbb{R}$, where, for $U \sim \mathcal{N}(0,1)$, and for every $x \in \mathbb{R}$, it is true that $\mathbb{E}\big\{4x^3U+4xU^3\big\} \equiv 0$. Thus, although f is not Lipschitz and not smooth on $\mathbb{R}$, it is still $\big(L_{\mathcal{F}}, \mathsf{D}, 4(\cdot)^3(\bullet)+4(\cdot)(\bullet)^3\big)$-SLipschitz on $\mathcal{F}$. Also note that, although f is indeed Lipschitz and has a Lipschitz gradient on $\mathcal{F}$ in the usual sense, these properties are not enough for our purposes; the fact that $u \in \mathbb{R}$ (and $\boldsymbol{u} \in \mathbb{R}^N$, in Definition 3) plays a key role in our analysis.

For an additional example of a SLipschitz function globally on $\mathbb{R}$ which is non-Lipschitz and non-smooth, even on compact subsets of $\mathbb{R}$, let $f(\cdot) \equiv \sqrt{|\cdot|}$. Of course, f is neither Lipschitz nor smooth on any subset $\mathcal{F} \subseteq \mathbb{R}$ containing the origin, compact or not. Still, for every $x \in \mathcal{F}$, the fact is that

$$\begin{aligned}\left|f(x+u)-f(x)\right| &\equiv \left|\sqrt{|x+u|}-\sqrt{|x|}\right|\\ &\leq \sqrt{\big|\,|x+u|-|x|\,\big|}\\ &\leq \sqrt{|u|},\end{aligned} \tag{146}$$

for all $u \in \mathbb{R}$. Therefore, although f is not Lipschitz and not smooth on $\mathcal{F}$, it is still $\big(1, \sqrt{|\cdot|}, 0\big)$-SLipschitz on $\mathcal{F}$. In this example, it is interesting to note that the function itself and the respective divergence actually coincide, that is, $f(\cdot) \equiv \sqrt{|\cdot|} \equiv \mathcal{D}(\cdot)$.

B Proof of Lemma 2

If $\mu \equiv 0$, the situation is trivial. So, for the rest of the proof, we assume that $\mu > 0$. Let $\mathcal{N} : \mathbb{R}^N \to \mathbb{R}$ be the standard Gaussian density on $\mathbb{R}^N$, that is,

$$\mathcal{N}(\boldsymbol{u}) \triangleq \frac{\exp\big(-\|\boldsymbol{u}\|_2^2/2\big)}{\sqrt{(2\pi)^N}}, \quad \boldsymbol{u} \in \mathbb{R}^N. \tag{147}$$

From condition (7), we have that

$$\begin{aligned}\infty &> \int \|\mu\boldsymbol{u}\|_2 \,|f(\mu\boldsymbol{u})|\,\mathcal{N}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\\ &\equiv \int \|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\,|f(\boldsymbol{x}+\mu\boldsymbol{u})|\,\mathcal{N}\left(\boldsymbol{u}+\frac{\boldsymbol{x}}{\mu}\right)\mathrm{d}\boldsymbol{u}\\ &\geq \mathcal{N}\left(\sqrt{2}\frac{\boldsymbol{x}}{\mu}\right)\int \|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\,|f(\boldsymbol{x}+\mu\boldsymbol{u})|\,\mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big)\mathrm{d}\boldsymbol{u},\end{aligned} \tag{148}$$

which implies that

$$\begin{aligned}\infty &> \int \|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\,|f(\boldsymbol{x}+\mu\boldsymbol{u})|\,\mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big)\mathrm{d}\boldsymbol{u}\\ &\equiv \int \|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\,|f(\boldsymbol{x}+\mu\boldsymbol{u})|\,\mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big)\mathbb{1}_{(0,1)}\left(\|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\right)\mathrm{d}\boldsymbol{u}\end{aligned}$$

$$+\int \|\boldsymbol{x}+\mu\boldsymbol{u}\|_2 \left|f\left(\boldsymbol{x}+\mu\boldsymbol{u}\right)\right| \mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big) \mathbb{1}_{[1,\infty)}\left(\|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\right) \mathrm{d}\boldsymbol{u}$$
$$\geq \int \left|f\left(\boldsymbol{x}+\mu\boldsymbol{u}\right)\right| \mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big) \mathbb{1}_{[1,\infty)}\left(\|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\right) \mathrm{d}\boldsymbol{u}. \tag{149}$$

Consequently,

$$\infty > \int \left|f\left(\boldsymbol{x}+\mu\boldsymbol{u}\right)\right| \mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big) \mathbb{1}_{[1,\infty)}\left(\|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\right) \mathrm{d}\boldsymbol{u}$$
$$+ \int \left|f\left(\boldsymbol{x}+\mu\boldsymbol{u}\right)\right| \mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big) \mathbb{1}_{(0,1)}\left(\|\boldsymbol{x}+\mu\boldsymbol{u}\|_2\right) \mathrm{d}\boldsymbol{u}$$
$$\equiv \int \left|f\left(\boldsymbol{x}+\mu\boldsymbol{u}\right)\right| \mathcal{N}\big(\sqrt{2}\boldsymbol{u}\big) \mathrm{d}\boldsymbol{u}, \tag{150}$$

from where it follows that the function $f\left(\boldsymbol{x}+\mu\boldsymbol{U}\left(\cdot\right)\right)$ is well-defined and in $\mathcal{Z}_1$, for all $\boldsymbol{x} \in \mathbb{R}^N$. Equivalently, we have shown that the function $f_\mu\left(\cdot\right) \equiv \mathbb{E}\left\{f\left(\left(\cdot\right)+\mu\boldsymbol{U}\right)\right\}$ is well-defined and finite, everywhere on $\mathbb{R}^N$. The rest of the first part, and the second part of Lemma 2 may be developed along the lines of [28], where we explicitly use the identity $\mathbb{E}\left\{\mathsf{T}\left(\boldsymbol{x},\boldsymbol{U}\right)\right\} \equiv 0$, for all $\boldsymbol{x} \in \mathcal{F}$, since T is a normal remainder on $\mathcal{F}$.

For the third part, the result on the existence and representation of ∇f_μ will follow by a careful application of the Dominated Convergence Theorem, which provides an *extension* of the standard Leibniz rule of Riemann integration, and permits interchangeability of differentiation and integration. Specifically, we will exploit a multidimensional version of ([12], Theorem 2.27). To this end, for $\mu > 0$, define for brevity

$$\varphi\left(\boldsymbol{x},\boldsymbol{u}\right) \triangleq f\left(\boldsymbol{u}\right)\mu^{-N}\mathcal{N}\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{\mu}\right), \quad \left(\boldsymbol{x},\boldsymbol{u}\right) \in \mathcal{F}\times\mathbb{R}^N. \tag{151}$$

By our construction, $\varphi\left(\boldsymbol{x},\cdot\right)$ is Lebesgue integrable on $\mathbb{R}^N$ for every $\boldsymbol{x} \in \mathbb{R}^N$, and $\varphi\left(\cdot,\boldsymbol{u}\right)$ is differentiable everywhere on $\mathbb{R}^N$ for every $\boldsymbol{u} \in \mathbb{R}^N$, with

$$\nabla_{\boldsymbol{x}}\varphi\left(\boldsymbol{x},\boldsymbol{u}\right) \equiv \mu^{-N-2} f\left(\boldsymbol{u}\right)\mathcal{N}\left(\frac{\boldsymbol{u}-\boldsymbol{x}}{\mu}\right)\left(\boldsymbol{u}-\boldsymbol{x}\right). \tag{152}$$

Now, consider any compact box $\mathcal{B} \subseteq \mathbb{R}^N$. We may write

$$\begin{aligned}\|\nabla_{\boldsymbol{x}}\varphi\left(\boldsymbol{x},\boldsymbol{u}\right)\|_2 &\equiv \mu^{-N-2}\mathcal{N}\left(\frac{\boldsymbol{u}-\boldsymbol{x}}{\mu}\right)\left|f\left(\boldsymbol{u}\right)\right|\|\boldsymbol{u}-\boldsymbol{x}\|_2 \\ &\leq \mu^{-N-2}\mathcal{N}\left(\frac{\boldsymbol{u}-\boldsymbol{x}}{\mu}\right)\left|f\left(\boldsymbol{u}\right)\right|\left(\|\boldsymbol{u}\|_2+\|\boldsymbol{x}\|_2\right) \\ &\leq \mu^{-N-2}\mathcal{N}\left(\frac{\boldsymbol{u}}{\mu}\right)\exp\left(\frac{\|\boldsymbol{u}\|_2\sup_{\boldsymbol{x}\in\mathcal{B}}\|\boldsymbol{x}\|_2}{\mu^2}\right)\left|f\left(\boldsymbol{u}\right)\right|\left(\|\boldsymbol{u}\|_2+\sup_{\boldsymbol{x}\in\mathcal{B}}\|\boldsymbol{x}\|_2\right) \\ &\triangleq \psi_{\mathcal{B}}\left(\boldsymbol{u}\right), \quad \forall \boldsymbol{u} \in \mathbb{R}^N. \end{aligned} \tag{153}$$

Note that, in the above, the use of the ℓ_2-norm is arbitrary; any (equivalent) vector norm works. Then, with $B \triangleq \sup_{\boldsymbol{x}\in\mathcal{B}}\|\boldsymbol{x}\|_2/\mu^2$, and via a simple change of variables, as in the beginning of the proof, it may be readily shown that $\psi_{\mathcal{B}}$ has a finite Lebesgue integral on $\mathbb{R}^N$ and thus it is true that, for every $\boldsymbol{u} \in \mathbb{R}^N$,

$$\sup_{\boldsymbol{x}\in\mathcal{B}}\|\nabla_{\boldsymbol{x}}\varphi\left(\boldsymbol{x},\boldsymbol{u}\right)\|_2 \leq \psi_{\mathcal{B}}\left(\boldsymbol{u}\right) \in \mathcal{L}_1\big(\mathbb{R}^N, \mathscr{B}\big(\mathbb{R}^N\big), \lambda; \mathbb{R}\big), \tag{154}$$

where $\lambda : \mathscr{B}(\mathbb{R}^M) \to \mathbb{R}_+$ denotes the corresponding Lebesgue measure. Then, it follows that the function $f_\mu(\cdot) \equiv \int \varphi(\cdot, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}$ is differentiable everywhere on $\mathcal{B}$, and that

$$\begin{aligned}
\nabla f_\mu(\boldsymbol{x}) \equiv \nabla \int \varphi(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u} &\equiv \int \nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \\
&\equiv \int \mu^{-N-2} f(\boldsymbol{u}) \mathcal{N}\left(\frac{\boldsymbol{u}-\boldsymbol{x}}{\mu}\right)(\boldsymbol{u}-\boldsymbol{x})\, \mathrm{d}\boldsymbol{u} \\
&= \int \mu^{-1} f(\boldsymbol{x}+\mu\boldsymbol{u}) \mathcal{N}(\boldsymbol{u})\boldsymbol{u} \mathrm{d}\boldsymbol{u} - \int \mu^{-1} f(\boldsymbol{x}) \mathcal{N}(\boldsymbol{u})\boldsymbol{u} \mathrm{d}\boldsymbol{u} \\
&\equiv \int \frac{f(\boldsymbol{x}+\mu\boldsymbol{u}) - f(\boldsymbol{x})}{\mu} \boldsymbol{u} \mathcal{N}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u},
\end{aligned} \tag{155}$$

for every $\boldsymbol{x} \in \mathcal{B}$ (Theorem 2.27 in [12]). But the box $\mathcal{B}$ is arbitrary, and any $\boldsymbol{x} \in \mathbb{R}^N$ is contained in a compact box. For the rest of the third part of Lemma 2, if f is $(L, \mathsf{D}, \mathsf{T})$-SLipschitz on $\mathcal{F}$, we may first write

$$\begin{aligned}
&\mathbb{E}\left\{\left\|\frac{f(\boldsymbol{x}+\mu\boldsymbol{U}) - f(\boldsymbol{x})}{\mu}\boldsymbol{U}\right\|_2^2\right\} \\
&\quad\equiv \frac{1}{\mu^2}\mathbb{E}\big\{|f(\boldsymbol{x}+\mu\boldsymbol{U}) - f(\boldsymbol{x})|^2 \|\boldsymbol{U}\|_2^2\big\} \\
&\quad\equiv \frac{1}{\mu^2}\mathbb{E}\big\{|f(\boldsymbol{x}+\mu\boldsymbol{U}) - f(\boldsymbol{x}) - \mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{U}) + \mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{U})|^2 \|\boldsymbol{U}\|_2^2\big\} \\
&\quad\le \frac{1}{\mu^2}\mathbb{E}\big\{\big(L\mathsf{D}(\mu\boldsymbol{U}) + |\mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{U})|\big)^2 \|\boldsymbol{U}\|_2^2\big\},
\end{aligned} \tag{156}$$

for all $\boldsymbol{x} \in \mathcal{F}$. Enough said. ■

Remark 1. Note that the conclusions of Lemma 2 regarding the gradient of f_μ *cannot* follow simply by the Leibniz rule, and this is due to the unbounded support of the Gaussian density, which makes the region of integration in the definition of f_μ the whole Euclidean space $\mathbb{R}^N$ (that is, infinite). ■

C Another Example of an Effective/Stable (D, T)-pair

To illustrate an additional case of an (uniformly) effective/stable (D, T)-pair and an efficient divergence, as well as its consequences on the conclusions of Lemma 2, let us consider the quadratic fit cost $f(\cdot) \equiv \|\boldsymbol{y} - \boldsymbol{A}(\cdot)\|_2^2$, for fixed and compatible $\boldsymbol{y} \in \mathbb{R}^{N'}$ and $\boldsymbol{A} \in \mathbb{R}^{N' \times N}$. Of course, f is smooth on $\mathbb{R}^N$ and therefore, we know already that it is SLipschitz on $\mathbb{R}^N$, as well. Still, at least for illustration, an associated (D, T)-pair may constructed in a more elaborate way for this example. First, for every pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^N \times \mathbb{R}^N$, it is true that

$$\begin{aligned}
&\|\boldsymbol{y} - \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{u})\|_2^2 - \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 \\
&\quad\equiv \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{u}\|_2^2 - 2\boldsymbol{y}^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{u} - \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \\
&\quad\equiv \boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{u} - 2(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{u} \\
&\quad\equiv \sum_{i,j} [\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}]_{i,j} u_i u_j - 2(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{u}
\end{aligned}$$

$$\equiv \sum_i \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,i} u_i^2 + 2 \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} u_i u_j - 2\big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big)^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{u}. \tag{157}$$

In other words,

$$\begin{aligned}
&\left| \|\boldsymbol{y} - \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{u})\|_2^2 - \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 - 2 \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} u_i u_j + 2\big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big)^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{u} \right| \\
&\quad \equiv \sum_i \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,i} u_i^2 \\
&\quad \le \max\left\{\mathrm{diag}\big(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\big)\right\} \|\boldsymbol{u}\|_2^2,
\end{aligned} \tag{158}$$

for all $(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^N \times \mathbb{R}^N$. We may then naturally define

$$L \triangleq \max\left\{\mathrm{diag}\big(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\big)\right\}, \tag{159}$$

$$\mathsf{D}(\boldsymbol{u}) \triangleq \|\boldsymbol{u}\|_2^2, \quad \forall \boldsymbol{u} \in \mathbb{R}^N \quad \text{and} \tag{160}$$

$$\mathsf{T}(\boldsymbol{x}, \boldsymbol{u}) \triangleq 2 \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} u_i u_j - 2\big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big)^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{u}, \quad \forall (\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{161}$$

where it is easy to see that $\mathbb{E}\{\mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{U})\} \equiv 0$, for all $\boldsymbol{x} \in \mathbb{R}^N$ and $\mu \ge 0$; thus, f is $(L, \mathsf{D}, \mathsf{T})$-SLipschitz on $\mathbb{R}^N$. Then, with $\mu_o \equiv 1$ (say), we may define, for every $(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\mathsf{D}(\mu\boldsymbol{u}) \equiv \mu^2 \mathsf{D}(\boldsymbol{u}) \triangleq \mu^2 \mathsf{d}(\boldsymbol{u}) \quad \text{and} \tag{162}$$

$$|\mathsf{T}(\boldsymbol{x}, \mu\boldsymbol{u})| \le \mu 2\left(\left| \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} u_i u_j \right| + \left| \big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big)^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{u} \right| \right) \triangleq \mu \mathsf{t}(\boldsymbol{x}, \boldsymbol{u}). \tag{163}$$

Also observe that both $\mathbb{E}\big\{\big(\mathsf{d}(\boldsymbol{U})\big)^2\big\}$ and $\mathbb{E}\big\{\big(\mathsf{d}(\boldsymbol{U})\big)^2 \|\boldsymbol{U}\|_2^2\big\}$ are finite. Additionally, we have

$$\begin{aligned}
\mathbb{E}\big\{\big(\mathsf{t}(\boldsymbol{x}, \boldsymbol{U})\big)^2 \|\boldsymbol{U}\|_2^2\big\} &\equiv \mathbb{E}\left\{ 4\left(\left| \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} U_i U_j \right| + \left| \big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big)^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{U} \right| \right)^2 \|\boldsymbol{U}\|_2^2 \right\} \\
&\le 8\mathbb{E}\left\{ \left(\sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} U_i U_j \right)^2 \|\boldsymbol{U}\|_2^2 \right\} + 8\mathbb{E}\big\{ \big((\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}} \boldsymbol{A}\boldsymbol{U} \big)^2 \|\boldsymbol{U}\|_2^2 \big\} \\
&\le 24N \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j}^2 + 8(N+2) \big\| \boldsymbol{A}^{\boldsymbol{T}} \big(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\big) \big\|_2^2 \\
&\equiv 24N \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j}^2 + 2(N+2) \|\nabla f(\boldsymbol{x})\|_2^2,
\end{aligned} \tag{164}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$, where we have used the facts that

$$\begin{aligned}
\mathbb{E}\left\{ \left(\sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} U_i U_j \right)^2 \|\boldsymbol{U}\|_2^2 \right\} &\equiv \sum_\tau \sum_{i,j,i>j} \sum_{k,l,k>l} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{k,l} \mathbb{E}\big\{ U_i U_j U_k U_l U_\tau^2 \big\} \\
&\equiv \sum_\tau \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j}^2 \mathbb{E}\big\{ U_i^2 U_j^2 U_\tau^2 \big\} \\
&\le 3N \sum_{i,j,i>j} \left[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\right]_{i,j}^2
\end{aligned} \tag{165}$$

and

$$\begin{aligned}\mathbb{E}\{((\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{U})^2\,\|\boldsymbol{U}\|_2^2\} &\equiv \mathbb{E}\{(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\,\|\boldsymbol{U}\|_2^2\}\\ &\equiv \mathbb{E}\{(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\,\|\boldsymbol{U}\|_2^2\,\boldsymbol{A}^{\boldsymbol{T}}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\}\\ &\equiv (\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\mathbb{E}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\,\|\boldsymbol{U}\|_2^2\}\boldsymbol{A}^{\boldsymbol{T}}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\\ &\equiv (\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{A}\sum_{\tau}\mathbb{E}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}U_\tau^2\}\boldsymbol{A}^{\boldsymbol{T}}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\\ &\equiv (N+2)\big\|\boldsymbol{A}^{\boldsymbol{T}}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\big\|_2^2, \end{aligned}\tag{166}$$

respectively, and where the last expression in (166) is due to the additional fact that

$$\begin{aligned}\sum_{\tau}\mathbb{E}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}U_\tau^2\} &\equiv \sum_{\tau}\mathbb{E}\begin{bmatrix} U_1^2U_\tau^2 & U_1U_2U_\tau^2 & \dots & U_1U_NU_\tau^2\\ & U_2^2U_\tau^2 & \dots & U_2U_NU_\tau^2\\ & & \ddots & \vdots\\ & \text{same here} & & U_{N-1}U_NU_\tau^2\\ & & & U_N^2U_\tau^2\end{bmatrix}\\ &\equiv \sum_{\tau}\begin{bmatrix} 1+2\mathbb{1}_{\{\tau\equiv 1\}} & 0 & \dots & 0\\ & 1+2\mathbb{1}_{\{\tau\equiv 2\}} & \dots & 0\\ & & \ddots & \vdots\\ & \text{same here} & & 0\\ & & & 1+2\mathbb{1}_{\{\tau\equiv N\}}\end{bmatrix}\\ &\equiv (N+2)\,\boldsymbol{I}_N. \end{aligned}\tag{167}$$

A similar procedure may be used to verify boundedness of $\mathbb{E}\{(\mathsf{t}(\cdot,\boldsymbol{U}))^2\}$. We thus see that the involved (D,T)-pair is 2-stable with $\mathsf{t}\equiv\mathsf{t}_2$ (2-effective, as well), and that D is efficient with $\varepsilon\equiv 1$.

Now, observe that f satisfies the moment condition (7) of Lemma 2. Consequently, this implies that

$$\sup_{\boldsymbol{x}\in\mathcal{F}}\big|f_\mu(\boldsymbol{x})-f(\boldsymbol{x})\big| \le \mu^2 L\mathbb{E}\{\|\boldsymbol{U}\|_2^2\} \equiv \mu^2 LN, \tag{168}$$

which of course means that the accuracy of the approximation increases superlinearly as μ decreases. Additionally, it follows that,

$$\begin{aligned}&\mathbb{E}\left\{\left\|\frac{f(\boldsymbol{x}+\mu\boldsymbol{U})-f(\boldsymbol{x})}{\mu}\boldsymbol{U}\right\|_2^2\right\}\\ &\le \frac{1}{\mu^2}\mathbb{E}\{(L\mathsf{D}(\mu\boldsymbol{U})+|\mathsf{T}(\boldsymbol{x},\mu\boldsymbol{U})|)^2\,\|\boldsymbol{U}\|_2^2\}\\ &\le \frac{1}{\mu^2}\mathbb{E}\{(L\mu^2\mathsf{d}(\boldsymbol{U})+\mu\mathsf{t}(\boldsymbol{x},\boldsymbol{U}))^2\,\|\boldsymbol{U}\|_2^2\}\\ &\le \mu^2 2L^2\mathbb{E}\{(\mathsf{d}(\boldsymbol{U}))^2\,\|\boldsymbol{U}\|_2^2\}+2\mathbb{E}\{(\mathsf{t}(\boldsymbol{x},\boldsymbol{U}))^2\,\|\boldsymbol{U}\|_2^2\}\\ &\equiv \mu^2 2L^2\mathbb{E}\{\|\boldsymbol{U}\|_2^4\,\|\boldsymbol{U}\|_2^2\}+2\mathbb{E}\{(\mathsf{t}(\boldsymbol{x},\boldsymbol{U}))^2\,\|\boldsymbol{U}\|_2^2\}\\ &\le \mu^2 2L^2(N+6)^3+\mu^2 48N\sum_{i,j,i>j}\big[\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}\big]_{i,j}^2+4(N+2)\,\|\nabla f(\boldsymbol{x})\|_2^2, \end{aligned}\tag{169}$$

yielding

$$\mathbb{E}\left\{\left\|\frac{f(\boldsymbol{x}+\mu\boldsymbol{U})-f(\boldsymbol{x})}{\mu}\boldsymbol{U}\right\|_2^2\right\}\le\mu^2\mathcal{O}\big(N^3\big)+\|\nabla f(\boldsymbol{x})\|_2^2\mathcal{O}(N),\tag{170}$$

for all $\boldsymbol{x}\in\mathbb{R}^N$. If, instead of $\mathbb{R}^N$, we consider any compact subset $\mathcal{F}\subseteq\mathbb{R}^N$, then our (D,T)-pair is uniformly 2-stable on $\mathcal{F}$, as well, by using the implication that $\sup_{\boldsymbol{x}\in\mathcal{X}}\|\nabla f(\boldsymbol{x})\|_2^2<\infty$.

D Specialization of Lemma 3 for Common Function Classes

It is possible to obtain more detailed and informative bounds than those in Lemma 3 by restricting our attention on classes of *random* cost functions satisfying Assumption 1 but where, additionally, the associated (D,T)-pair is known explicitly.

D.1 Lipschitz Class on $\mathbb{R}^N$

The first class we would like to discuss is that consisting of functions satisfying the *global* Lipschitz-like condition

$$\|F(\boldsymbol{x}_1,\boldsymbol{W})-F(\boldsymbol{x}_2,\boldsymbol{W})\|_{\mathcal{L}_2}\le G\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2,\quad\forall(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^N\times\mathbb{R}^N.\tag{171}$$

Of course, (171) implies that the expected cost $\mathbb{E}\{F(\cdot,\boldsymbol{W})\}$ is (*globally*) G-Lipschitz on $\mathbb{R}^N$. By setting $\boldsymbol{u}\equiv\boldsymbol{x}_1-\boldsymbol{x}_2\in\mathbb{R}^N$, it easily follows that the choices $\mathsf{D}(\cdot)\equiv\|\cdot\|_2$ and $\mathsf{T}\equiv 0$ are valid, and that the particular (D,T)-pair is trivially uniformly 2-effective on $\mathcal{X}$, with $\mathsf{d}(\cdot)\equiv\|\cdot\|_2$ and $\mathsf{t}_2\equiv 0$. Then, the following corollary to Lemma 3 may be formulated. The proof is omitted.

Corollary 1. (Smoothed Convex Surrogates | Lipschitz Class) *Suppose that condition (171) is true. Then, conditions* **C1** *and* **C3** *are satisfied automatically. If, additionally, the rest of Assumption 1 is in effect, then Lemma 3 implies that*

$$\sup_{\boldsymbol{x}\in\mathcal{X}}|\phi_\mu(\boldsymbol{x})-\phi(\boldsymbol{x})|\le\mu G\sqrt{N}+c\mathcal{C}(\mu)\big(2\mu G\sqrt{N}+\mu\big),\tag{172}$$

where

$$\mathcal{C}(\mu)\le\mathbb{1}_{\{p\equiv1\}}+\eta^{-p/2}\big(\mathcal{R}(0)+2V+2\mu G\sqrt{N}+\mu\big)^{p/2}\mathbb{1}_{\{p\in(1,2]\}}.\tag{173}$$

D.2 Smooth Class on $\mathbb{R}^N$

The second class under the microscope consists of functions obeying the *global* smoothness-like condition

$$\|F(\boldsymbol{x}_1,\boldsymbol{W})-F(\boldsymbol{x}_2,\boldsymbol{W})-(\nabla F(\boldsymbol{x}_2,\boldsymbol{W}))^{\boldsymbol{T}}(\boldsymbol{x}_1-\boldsymbol{x}_2)\|_{\mathcal{L}_2}\le G\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2^2,\tag{174}$$

for all $(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^N\times\mathbb{R}^N$. Again, (174) implies that the expected cost $\mathbb{E}\{F(\cdot,\boldsymbol{W})\}$ is (*globally*) $2G$-smooth on $\mathbb{R}^N$. In this case, setting $\boldsymbol{u}\equiv\boldsymbol{x}_1-\boldsymbol{x}_2\in\mathbb{R}^N$ yields

$$\sup_{\boldsymbol{x}\in\mathcal{X}}\|F(\boldsymbol{x}+\boldsymbol{u},\boldsymbol{W})-F(\boldsymbol{x},\boldsymbol{W})-(\nabla F(\boldsymbol{x},\boldsymbol{W}))^{\boldsymbol{T}}\boldsymbol{u}\|_{\mathcal{L}_2}\le G\|\boldsymbol{u}\|_2^2,\quad\forall\boldsymbol{u}\in\mathbb{R}^N.\tag{175}$$

It is then natural to choose $\mathsf{D}(\cdot) \equiv \|\cdot\|_2^2$ and $\mathsf{T}([\bullet,\star],\cdot) \equiv (\nabla F(\bullet,\star))^{\boldsymbol{T}}(\cdot)$ as the associated (D,T)-pair. Additionally, as it might be evident at this point it is true that the previously defined (D,T)-pair is uniformly 2-effective on $\mathcal{X}$, whenever $\mathcal{X}$ is a compact subset of $\mathbb{R}^N$. Indeed, we may choose $\mathsf{d}(\cdot) \equiv \|\cdot\|_2^2$ and $\mathsf{t}_2(\bullet,\cdot) \equiv \|(\nabla F(\bullet,\boldsymbol{W}))^{\boldsymbol{T}}(\cdot)\|_{\mathcal{L}_2}$, where we have, for every $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X} \times \mathbb{R}^N$,

$$
\begin{aligned}
\mathsf{t}_2(\boldsymbol{x},\boldsymbol{u}) &\equiv \sqrt{\boldsymbol{u}^{\boldsymbol{T}}\mathbb{E}\{\nabla F(\boldsymbol{x},\boldsymbol{W})(\nabla F(\boldsymbol{x},\boldsymbol{W}))^{\boldsymbol{T}}\}\boldsymbol{u}} \\
&\triangleq \sqrt{\boldsymbol{u}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{u}}.
\end{aligned}
\tag{176}
$$

Therefore, it us true that, for every $\boldsymbol{x} \in \mathcal{X}$,

$$
\begin{aligned}
\|\mathsf{t}_2(\boldsymbol{x},\boldsymbol{U})\|_{\mathcal{L}_2} &\equiv \sqrt{\mathbb{E}\{\boldsymbol{U}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{U}\}} \\
&\equiv \sqrt{\mathbb{E}\{\mathrm{tr}\{\boldsymbol{U}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\boldsymbol{U}\}\}} \\
&\equiv \sqrt{\mathbb{E}\{\mathrm{tr}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\nabla\mathsf{F}(\boldsymbol{x})\}\}} \\
&\equiv \sqrt{\mathrm{tr}\{\mathbb{E}\{\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\}\nabla\mathsf{F}(\boldsymbol{x})\}} \\
&\equiv \sqrt{\mathrm{tr}\{\nabla\mathsf{F}(\boldsymbol{x})\}} \\
&\equiv \big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_2},
\end{aligned}
\tag{177}
$$

and, as a result, $\mathcal{T}_2 < \infty$, under our assumptions. Note that, in this case, $\varepsilon \equiv 1$, as in Appendix C. Given the discussion above, we may formulate another corollary to Lemma 3, regarding the case of the smooth class. The main arguments of the proof have already been presented.

Corollary 2. (Smoothed Convex Surrogates | Smooth Class) *Suppose that condition (174) is true. Then, condition* **C1** *is satisfied automatically and, whenever* $\sup_{\boldsymbol{x}\in\mathcal{X}}\big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_2} < \infty$, **C3** *is satisfied, as well. If, moreover, the rest of Assumption 1 is in effect, then Lemma 3 implies that*

$$
\sup_{\boldsymbol{x}\in\mathcal{X}} |\phi_\mu(\boldsymbol{x}) - \phi(\boldsymbol{x})| \le \mu^2 GN + c\mathcal{C}(\mu)\big(6\mu^2 GN + \mu(\mathcal{T}_2+1)\big),
\tag{178}
$$

where

$$
\mathcal{C}(\mu) \le \mathbb{1}_{\{p\equiv 1\}} + \eta^{-p/2}\big(\mathcal{R}(0) + 2V + 6\mu^2 GN + \mu(\mathcal{T}_2+1)\big)^{p/2}\mathbb{1}_{\{p\in(1,2]\}},
\tag{179}
$$

with $\mathcal{T}_2$ *being an intrinsic feature of* F*. Specifically, we have* $\mathcal{T}_2 \equiv \sup_{\boldsymbol{x}\in\mathcal{X}}\big\|\|\nabla F(\boldsymbol{x},\boldsymbol{W})\|_2\big\|_{\mathcal{L}_2}$.

E Recursions: Proofs

E.1 Proof of Lemma 6

Fix $n \in \mathbb{N}$ and let $p > 1$; if $p \equiv 1$ the derivation is similar. Under Assumption 2, by nonexpansiveness of the projection operator onto the closed and convex set $\mathcal{X}$, and by the triangle inequality,

$$
\begin{aligned}
\|\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\|_2 &\le \alpha_n\big\|\widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n, y^n, z^n)\big\|_2 \\
&\le \alpha_n\big\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1}\big\|_2 + \alpha_n c\big\|\boldsymbol{\Delta}_{\mu,p}^{n+1}(\boldsymbol{x}^n, y^n, z^n)\big\|_2,
\end{aligned}
\tag{180}
$$

where $\boldsymbol{\Delta}_{\mu,p}^{n+1}$ may be further expanded as

$$\begin{aligned}
&\left\|\boldsymbol{\Delta}_{\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}, z^{n}\right)\right\|_{2} \\
&\quad \equiv \left\|\left(z^{n}\right)^{\frac{1-p}{p}}\left(\boldsymbol{U}_{2}^{n+1}+\Delta_{1,\mu}^{n+1}\left(\boldsymbol{x}^{n}\right) \boldsymbol{U}_{1}^{n+1} U^{n+1}\right) \Delta_{2,\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}\right)\right\|_{2} \\
&\quad \leq \eta^{1-p}\left(\left\|\Delta_{2,\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}\right) \boldsymbol{U}_{2}^{n+1}\right\|_{2}+\left|\Delta_{2,\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}\right) U^{n+1}\right|\left\|\Delta_{1,\mu}^{n+1}\left(\boldsymbol{x}^{n}\right) \boldsymbol{U}_{1}^{n+1}\right\|_{2}\right). \qquad (181)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^{n}\right\|_{2} \leq\, & \alpha_{n}\left\|\Delta_{1,\mu}^{n+1}\left(\boldsymbol{x}^{n}\right) \boldsymbol{U}_{1}^{n+1}\right\|_{2}+\alpha_{n} c \eta^{1-p}\left|\Delta_{2,\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}\right) U^{n+1}\right|\left\|\Delta_{1,\mu}^{n+1}\left(\boldsymbol{x}^{n}\right) \boldsymbol{U}_{1}^{n+1}\right\|_{2} \\
&+\alpha_{n} c \eta^{1-p}\left\|\Delta_{2,\mu,p}^{n+1}\left(\boldsymbol{x}^{n}, y^{n}\right) \boldsymbol{U}_{2}^{n+1}\right\|_{2}, \qquad (182)
\end{aligned}$$

almost everywhere relative to $\mathcal{P}$. Now, for $(\boldsymbol{x}, y) \in \mathcal{X} \times \mathcal{Y}$, it is true that $\mathbb{E}\left\{\left\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}) \boldsymbol{U}_{1}^{n+1}\right\|_{2}^{2}\right\} \leq B_{1}$, $\mathbb{E}\left\{\left\|\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}, y) \boldsymbol{U}_{2}^{n+1}\right\|_{2}^{2}\right\} \leq B_{2}$, and also

$$\begin{aligned}
&\mathbb{E}\left\{\left(\left|U^{n+1} \Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}, y)\right|\left\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}) \boldsymbol{U}_{1}^{n+1}\right\|_{2}\right)^{2}\right\} \\
&\quad \equiv \mathbb{E}\left\{\left|\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}, y) U^{n+1}\right|^{2}\right\} \mathbb{E}\left\{\left\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}) \boldsymbol{U}_{1}^{n+1}\right\|_{2}^{2}\right\} \leq B_{1} B_{2}. \qquad (183)
\end{aligned}$$

It is then true that

$$\begin{aligned}
&\Big\| \alpha_{n}\left\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}) \boldsymbol{U}_{1}^{n+1}\right\|_{2}+\alpha_{n} c \eta^{1-p}\left|\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}, y) U^{n+1}\right|\left\|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}) \boldsymbol{U}_{1}^{n+1}\right\|_{2} \\
&\qquad +\alpha_{n} c \eta^{1-p}\left\|\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}, y) \boldsymbol{U}_{2}^{n+1}\right\|_{2}\Big\|_{\mathcal{L}_{2}} \\
&\quad \leq \alpha_{n} \sqrt{B_{1}}+\alpha_{n} c \eta^{1-p} \sqrt{B_{1} B_{2}}+\alpha_{n} c \eta^{1-p} \sqrt{B_{2}} \\
&\quad \equiv \alpha_{n}\left(\sqrt{B_{1}}+c \eta^{1-p}\left(\sqrt{B_{1} B_{2}}+\sqrt{B_{2}}\right)\right), \qquad (184)
\end{aligned}$$

for all $(\boldsymbol{x}, y) \in \mathcal{X} \times \mathcal{Y}$. Therefore, taking squares and conditional expectations relative to $\mathscr{D}^{n}$ on both sides of (182) yields

$$\mathbb{E}_{\mathscr{D}^{n}}\left\{\left\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^{n}\right\|_{2}^{2}\right\} \leq \alpha_{n}^{2}\left(\sqrt{B_{1}}+c \eta^{1-p}\left(\sqrt{B_{1} B_{2}}+\sqrt{B_{2}}\right)\right)^{2} \triangleq \Sigma_{p}^{1} \alpha_{n}^{2}, \qquad (185)$$

almost everywhere relative to $\mathcal{P}$. But $\mathbb{N}$ is countable. ■

E.2 Proof of Lemma 7

Fix $n \in \mathbb{N}$, and let $y^{n}-s_{\mu}\left(\boldsymbol{x}^{n}\right) \triangleq E_{s}^{n}$, for brevity. By nonexpansiveness of Euclidean projections and by adding and subtracting the term $s_{\mu}\left(\boldsymbol{x}^{n}\right)$, it follows that

$$\begin{aligned}
\left|E_{s}^{n+1}\right|^{2} \leq\, & \left(1-\beta_{n}\right)\left|E_{s}^{n}\right|^{2}+2 \beta_{n}^{2}\left|F\left(\boldsymbol{x}^{n}+\mu \boldsymbol{U}_{1}^{n+1}, \boldsymbol{W}_{1}^{n+1}\right)-s_{\mu}\left(\boldsymbol{x}^{n}\right)\right|^{2} \\
&+2\left(1+\beta_{n}\right)\left(1-\beta_{n}\right) \beta_{n} E_{s}^{n}\left(F\left(\boldsymbol{x}^{n}+\mu \boldsymbol{U}_{1}^{n+1}, \boldsymbol{W}_{1}^{n+1}\right)-s_{\mu}\left(\boldsymbol{x}^{n}\right)\right) \\
&+2 \beta_{n}^{-1}\left|s_{\mu}\left(\boldsymbol{x}^{n+1}\right)-s_{\mu}\left(\boldsymbol{x}^{n}\right)\right|_{2}^{2}, \qquad (186)
\end{aligned}$$

where we have used our assumption that $\beta_{n} \leq 1$. Taking expectations relative to $\mathscr{D}^{n}$ on both sides, and by Proposition 2 and Lemma 5, we have

$$\mathbb{E}_{\mathscr{D}^{n}}\left\{\left|E_{s}^{n+1}\right|^{2}\right\} \leq\left(1-\beta_{n}\right)\left|E_{s}^{n}\right|^{2}+\beta_{n}^{2} 2\left(V_{1}^{\prime}\right)^{2}+0+\beta_{n}^{-1} 2 B_{1} \mathbb{E}_{\mathscr{D}^{n}}\left\{\left\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^{n}\right\|_{2}^{2}\right\}, \qquad (187)$$

almost everywhere relative to $\mathcal{P}$. Lemma 6 and the fact that $\mathbb{N}$ is countable complete the proof. ■

E.3 Proof of Lemma 8

Fix $n \in \mathbb{N}$. As in the proof of Lemma 7, by nonexpansiveness of projections and by adding and subtracting appropriate terms, it is easy to show that $z^n - g_\mu(\boldsymbol{x}^n, y^n) \triangleq E_g^n$ may be written as

$$E_g^{n+1} \equiv (1-\gamma_n) E_g^n + \big(g_\mu(\boldsymbol{x}^n, y^n) - g_\mu(\boldsymbol{x}^{n+1}, y^{n+1})\big) \\ + \gamma_n \big(\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big). \tag{188}$$

Then, it is true that

$$\big|E_g^{n+1}\big|^2 \equiv (1+\gamma_n)(1-\gamma_n)^2 \big|E_g^n\big|^2 \\ + (1+\gamma_n)\gamma_n^2 \big|\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big|^2 \\ + 2\big(1-\gamma_n^2\big)\gamma_n E_g^n \big(\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big) \\ + \big(1+\gamma_n^{-1}\big)\big|g_\mu(\boldsymbol{x}^n, y^n) - g_\mu(\boldsymbol{x}^{n+1}, y^{n+1})\big|^2. \tag{189}$$

Taking conditional expectations relative to $\mathscr{D}^n$ on both sides and for $\gamma_n \le 1$, we get

$$\mathbb{E}_{\mathscr{D}^n}\big\{\big|E_g^{n+1}\big|^2\big\} \le (1-\gamma_n)\big|E_g^n\big|^2 \\ + 2\gamma_n^2 \mathbb{E}_{\mathscr{D}^n}\big\{\big|\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big|^2\big\} \\ + 2\gamma_n^{-1} \mathbb{E}_{\mathscr{D}^n}\big\{\big|g_\mu(\boldsymbol{x}^n, y^n) - g_\mu(\boldsymbol{x}^{n+1}, y^{n+1})\big|^2\big\} + 0, \tag{190}$$

almost everywhere relative to $\mathcal{P}$. Next, for the first of the last two nonzero terms on the right-hand side of (190), we may write

$$\mathbb{E}_{\mathscr{D}^n}\big\{\big|\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big|^2\big\} \\ \le \mathbb{E}_{\mathscr{D}^n}\big\{\big|\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p\big|^2\big\} \\ \equiv \mathbb{E}_{\mathscr{D}^n}\big\{\mathbb{E}_{\mathscr{D}^n, \boldsymbol{U}_2^{n+1}, U^{n+1}}\big\{\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^{2p}\big\}\big\}, \tag{191}$$

almost everywhere relative to $\mathcal{P}$. Now, for every $(\boldsymbol{x}, y) \in \mathcal{X} \times \mathcal{Y}$, already in the proof of Lemma 5 we have used the fact that

$$\mathbb{E}\big\{\big(\mathcal{R}\big(F\big(\boldsymbol{x} + \mu \boldsymbol{u}, \boldsymbol{W}\big) - \mu u - y\big)\big)^{2p}\big\} \\ \equiv \|(\mathcal{R}(F(\boldsymbol{x} + \mu \boldsymbol{u}, \boldsymbol{W}) - (y + \mu u)))\|_{\mathcal{L}_{2p}}^{2p} \\ \le \big(\mathcal{R}(0) + 2V_p + \mu^{1+\varepsilon} G \mathcal{D}_1 + \mu^{1+\varepsilon} G \mathsf{d}(\boldsymbol{u}) + \mu \mathsf{t}_4(\boldsymbol{x}, \boldsymbol{u}) + \mu |u|\big)^{2p}, \tag{192}$$

implying the existence of some constant $\overline{\Sigma}_1 < \infty$, problem dependent but independent of $\boldsymbol{x}$ and y (due to condition $\overline{\textbf{C3}}$) and increasing and bounded in μ, such that

$$\mathbb{E}_{\mathscr{D}^n}\big\{\big|\big(\mathcal{R}\big(F\big(\boldsymbol{x}^n + \mu \boldsymbol{U}_2^{n+1}, \boldsymbol{W}_2^{n+1}\big) - \mu U^{n+1} - y^n\big)\big)^p - g_\mu(\boldsymbol{x}^n, y^n)\big|^2\big\} \le \overline{\Sigma}_1. \tag{193}$$

almost everywhere relative to $\mathcal{P}$. For the second of the last two nonzero terms on the right-hand side of (190), observe that, by Lemma 5, we immediately obtain

$$\big|g_\mu(\boldsymbol{x}^{n+1}, y^{n+1}) - g_\mu(\boldsymbol{x}^n, y^n)\big|$$

$$
\begin{aligned}
&\equiv \sqrt{B_2}\sqrt{\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^n\|_2^2+|y^{n+1}-y^n|^2}\\
&\le \sqrt{B_2}\big(\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^n\|_2+|y^{n+1}-y^n|\big)\\
&\le \sqrt{B_2}\big(\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^n\|_2+\beta_n\big|F\big(\boldsymbol{x}^n+\mu\boldsymbol{U}_1^{n+1},\boldsymbol{W}_1^{n+1}\big)-y^n\big|\big), \qquad (194)
\end{aligned}
$$

yielding, by (say) Proposition 2 (and condition **C4**)

$$
\begin{aligned}
&\mathbb{E}_{\mathscr{D}^n}\big\{\big|g_\mu(\boldsymbol{x}^n,y^n)-g_\mu(\boldsymbol{x}^{n+1},y^{n+1})\big|^2\big\}\\
&\le 2B_2\mathbb{E}_{\mathscr{D}^n}\big\{\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^n\|_2^2\big\}+\beta_n^2 2B_2\mathbb{E}_{\mathscr{D}^n}\big\{\big|F\big(\boldsymbol{x}^n+\mu\boldsymbol{U}_1^{n+1},\boldsymbol{W}_1^{n+1}\big)-y^n\big|^2\big\}\\
&\le 2B_2\mathbb{E}_{\mathscr{D}^n}\big\{\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^n\|_2^2\big\}+\beta_n^2 4B_2\big(\mathbb{E}_{\mathscr{D}^n}\big\{\big|F\big(\boldsymbol{x}^n+\mu\boldsymbol{U}_1^{n+1},\boldsymbol{W}_1^{n+1}\big)\big|^2\big\}+(y^n)^2\big)\\
&\le 2B_2\Sigma_p^1\alpha_n^2+\beta_n^2 4B_2\big((V_1')^2+(\mu^{1+\varepsilon}G\mathcal{D}_1+V)^2\big), \qquad (195)
\end{aligned}
$$

almost everywhere relative to $\mathcal{P}$. Combining (195), (193) and (190), we end up with the desired inequality, being valid almost everywhere relative to $\mathcal{P}$. But $\mathbb{N}$ is countable. ■

E.4 Proof of Lemma 9

As usual, fix $n\in\mathbb{N}^+$, and let $p>1$; again, if $p\equiv 1$ the derivation is similar. Nonexpansiveness of the projection operator onto $\mathcal{X}$ yields

$$
\begin{aligned}
\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^o\|_2^2 &\le \big\|\boldsymbol{x}^n-\boldsymbol{x}^o-\alpha_n\widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n,y^n,z^n)\big\|_2^2\\
&\equiv \|\boldsymbol{x}^n-\boldsymbol{x}^o\|_2^2+\alpha_n^2\big\|\widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n,y^n,z^n)\big\|_2^2\\
&\quad -2\alpha_n(\boldsymbol{x}^n-\boldsymbol{x}^o)^{\boldsymbol{T}}\widehat{\nabla}_\mu^{n+1}\phi\big(\boldsymbol{x}^n,s_\mu(\boldsymbol{x}^n),g_\mu'(\boldsymbol{x}^n)\big)+\boldsymbol{U}^{n+1}, \qquad (196)
\end{aligned}
$$

where $\boldsymbol{U}^{n+1}:\Omega\to\mathbb{R}$ is defined as

$$
\begin{aligned}
\boldsymbol{U}^{n+1} &\triangleq 2c\alpha_n(\boldsymbol{x}^n-\boldsymbol{x}^o)^{\boldsymbol{T}}\big(\boldsymbol{\Delta}_{\mu,p}^{n+1}\big(\boldsymbol{x}^n,s_\mu(\boldsymbol{x}^n),g_\mu'(\boldsymbol{x}^n)\big)-\boldsymbol{\Delta}_{\mu,p}^{n+1}(\boldsymbol{x}^n,y^n,z^n)\big)\\
&\equiv 2c\alpha_n(\boldsymbol{x}^n-\boldsymbol{x}^o)^{\boldsymbol{T}}\\
&\quad\times\Bigg(\big(g_\mu'(\boldsymbol{x}^n)\big)^{(1-p)/p}\Big[\boldsymbol{I}_N\Big|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1}\Big]\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}^n,s_\mu(\boldsymbol{x}^n))\begin{bmatrix}\boldsymbol{U}_2^{n+1}\\ U^{n+1}\end{bmatrix}\\
&\qquad -(z^n)^{(1-p)/p}\Big[\boldsymbol{I}_N\Big|\Delta_{1,\mu}^{n+1}(\boldsymbol{x}^n)\boldsymbol{U}_1^{n+1}\Big]\Delta_{2,\mu,p}^{n+1}(\boldsymbol{x}^n,y^n)\begin{bmatrix}\boldsymbol{U}_2^{n+1}\\ U^{n+1}\end{bmatrix}\Bigg). \qquad (197)
\end{aligned}
$$

From the proof of Lemma 6, it follows that

$$
\mathbb{E}_{\mathscr{D}^n}\big\{\big\|\widehat{\nabla}_\mu^{n+1}\phi(\boldsymbol{x}^n,y^n,z^n)\big\|_2^2\big\}\le\Sigma_p^1, \qquad (198)
$$

almost everywhere relative to $\mathcal{P}$. Hence, taking conditional expectations on both sides of (196) relative to $\mathscr{D}^n$ and exploiting the convexity of ϕ_μ, we have

$$
\begin{aligned}
\mathbb{E}_{\mathscr{D}^n}\big\{\|\boldsymbol{x}^{n+1}-\boldsymbol{x}^o\|_2^2\big\} &\le \|\boldsymbol{x}^n-\boldsymbol{x}^o\|_2^2+\Sigma_p^1\alpha_n^2-2\alpha_n(\boldsymbol{x}^n-\boldsymbol{x}^o)^{\boldsymbol{T}}\nabla\phi_\mu(\boldsymbol{x}^n)+\mathbb{E}_{\mathscr{D}^n}\big\{\boldsymbol{U}^{n+1}\big\}\\
&\le \|\boldsymbol{x}^n-\boldsymbol{x}^o\|_2^2+\Sigma_p^1\alpha_n^2-2\alpha_n\big(\phi_\mu(\boldsymbol{x}^n)-\phi_\mu^o\big)+\mathbb{E}_{\mathscr{D}^n}\big\{\boldsymbol{U}^{n+1}\big\}, \qquad (199)
\end{aligned}
$$

almost everywhere relative to $\mathcal{P}$.

Let us focus on the residual term $\mathbb{E}_{\mathscr{D}^n}\{\boldsymbol{U}^{n+1}\}$. By construction, we have, for every $(\boldsymbol{x}, y) \in \mathcal{X} \times \mathcal{Y}$,

$$\begin{aligned}
&\mathbb{E}\Big\{\Big[\boldsymbol{I}_N \Big| \Delta^{n+1}_{1,\mu}(\boldsymbol{x})\, \boldsymbol{U}^{n+1}_1\Big]\, \Delta^{n+1}_{2,\mu,p}(\boldsymbol{x}, y) \begin{bmatrix} \boldsymbol{U}^{n+1}_2 \\ U^{n+1} \end{bmatrix}\Big\} \\
&\quad \equiv \mathbb{E}\Big\{\Big[\boldsymbol{I}_N \Big| \Delta^{n+1}_{1,\mu}(\boldsymbol{x})\, \boldsymbol{U}^{n+1}_1\Big]\Big\} \mathbb{E}\Big\{\Delta^{n+1}_{2,\mu,p}(\boldsymbol{x}, y) \begin{bmatrix} \boldsymbol{U}^{n+1}_2 \\ U^{n+1} \end{bmatrix}\Big\} \\
&\quad \equiv \big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x})\big]\, \nabla g_\mu(\boldsymbol{x}, y).
\end{aligned} \tag{200}$$

Consequently, by Cauchy-Schwarz, it is true that

$$\begin{aligned}
\mathbb{E}_{\mathscr{D}^n}\{\boldsymbol{U}^{n+1}\} &\equiv 2c\alpha_n (\boldsymbol{x}^n - \boldsymbol{x}^o)^{\boldsymbol{T}} \Big(\big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} \big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big]\, \nabla g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n)) \\
&\qquad - (z^n)^{(1-p)/p} \big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big]\, \nabla g_\mu(\boldsymbol{x}^n, y^n)\Big) \\
&\equiv 2c\alpha_n (\boldsymbol{x}^n - \boldsymbol{x}^o)^{\boldsymbol{T}} \big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big] \\
&\qquad \times \Big(\nabla g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n)) \big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} - \nabla g_\mu(\boldsymbol{x}^n, y^n)(z^n)^{(1-p)/p}\Big) \\
&\le 2c\alpha_n \|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2 \big\|\big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big]\big\|_2 \\
&\qquad \times \big\| \nabla g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n)) \big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} - \nabla g_\mu(\boldsymbol{x}^n, y^n)(z^n)^{(1-p)/p}\big\|_2,
\end{aligned} \tag{201}$$

almost everywhere relative to $\mathcal{P}$. We may further bound the last term (left to right) on the right-hand side of (201) from above as

$$\begin{aligned}
&\big\| \nabla g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n)) \big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} - \nabla g_\mu(\boldsymbol{x}^n, y^n)(z^n)^{(1-p)/p}\big\|_2 \\
&\quad \le \big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} \big\|\nabla g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n)) - \nabla g_\mu(\boldsymbol{x}^n, y^n)\big\|_2 \\
&\qquad + \big\|\nabla g_\mu(\boldsymbol{x}^n, y^n)\big\|_2 \big|\big(g'_\mu(\boldsymbol{x}^n)\big)^{(1-p)/p} - (z^n)^{(1-p)/p}\big| \qquad (202) \\
&\quad \le \eta^{(1-p)} L |y^n - s_\mu(\boldsymbol{x}^n)| + \frac{p-1}{p} \eta^{(1-2p)} \sqrt{B_2} |z^n - g'_\mu(\boldsymbol{x}^n)| \\
&\quad \le \overline{\Sigma}_2 \big(|y^n - s_\mu(\boldsymbol{x}^n)| + |z^n - g'_\mu(\boldsymbol{x}^n)|\big), \qquad (203)
\end{aligned}$$

almost everywhere relative to $\mathcal{P}$, where $\overline{\Sigma}_2 \triangleq \eta^{(1-p)} \max\{L, p^{-1}(p-1)\eta^{-p}\sqrt{B_2}\}$. And since

$$\big\|\big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big]\big\|_2 \le \big\|\big[\boldsymbol{I}_N \big| \nabla s_\mu(\boldsymbol{x}^n)\big]\big\|_F \le \sqrt{N + \|\nabla s_\mu(\boldsymbol{x}^n)\|_2^2} \le \sqrt{N + B_1}, \tag{204}$$

we may also write

$$\mathbb{E}_{\mathscr{D}^n}\{\boldsymbol{U}^{n+1}\} \le 2\big(\sqrt{N + B_1}\big) \overline{\Sigma}_2 c\alpha_n \|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2 \big(|y^n - s_\mu(\boldsymbol{x}^n)| + |z^n - g'_\mu(\boldsymbol{x}^n)|\big), \tag{205}$$

almost everywhere relative to $\mathcal{P}$. As a next step, by Lemma 5, we observe that

$$\begin{aligned}
|z^n - g'_\mu(\boldsymbol{x}^n)| &\le |z^n - g_\mu(\boldsymbol{x}^n, y^n)| + |g_\mu(\boldsymbol{x}^n, y^n) - g_\mu(\boldsymbol{x}^n, s_\mu(\boldsymbol{x}^n))| \\
&\le |z^n - g_\mu(\boldsymbol{x}^n, y^n)| + \sqrt{B_2} |y^n - s_\mu(\boldsymbol{x}^n)|.
\end{aligned} \tag{206}$$

This yields, in turn,

$$\mathbb{E}_{\mathscr{D}^n}\{\boldsymbol{U}^{n+1}\} \le 2\overline{\Sigma}_3 c\alpha_n \|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2 \big(|y^n - s_\mu(\boldsymbol{x}^n)| + |z^n - g_\mu(\boldsymbol{x}^n, y^n)|\big), \tag{207}$$

almost everywhere relative to $\mathcal{P}$, where $\overline{\Sigma}_3 \triangleq \big(\sqrt{N+B_1}\big)(1+\sqrt{B_2})\overline{\Sigma}_2$. Introducing the stepsizes β_n and γ_n, $\mathbb{E}_{\mathscr{D}^n}\big\{\boldsymbol{U}^{n+1}\big\}$ may be further bounded as

$$\mathbb{E}_{\mathscr{D}^n}\big\{\boldsymbol{U}^{n+1}\big\} \leq \overline{\Sigma}_3^2 c^2 \frac{\alpha_n^2}{\beta_n} \|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2 + \beta_n |y^n - s_\mu(\boldsymbol{x}^n)|^2 + \overline{\Sigma}_3^2 c^2 \frac{\alpha_n^2}{\gamma_n} \|\boldsymbol{x}^n - \boldsymbol{x}^o\|_2^2 + \gamma_n |z^n - g_\mu(\boldsymbol{x}^n, y^n)|^2, \tag{208}$$

almost everywhere relative to $\mathcal{P}$. Calling $\Sigma_p^4 \equiv \overline{\Sigma}_3^2$ whenever $p > 1$ (and accordingly whenever $p \equiv 1$), combining (208) with (199), and the fact that $\mathbb{N}$ is countable completes the proof. ■

References

[1] Prashanth L. A. and Michael Fu. Risk-Sensitive Reinforcement Learning: A Constrained Optimization Viewpoint. *arXiv preprint, arXiv:1810.09126*, October 2018.

[2] Shabbir Ahmed, Ulaş Çakmak, and Alexander Shapiro. Coherent Risk Measures in Inventory Problems. *European Journal of Operational Research*, 182(1):226–238, October 2007. ISSN 03772217. doi: 10.1016/j.ejor.2006.07.016.

[3] Robert B. Ash and Catherine Doléans-Dade. *Probability and Measure Theory*. Academic Press, 2000. ISBN 0120652021. doi: 10.2307/2291440.

[4] Krishnakumar Balasubramanian and Saeed Ghadimi. Zeroth-order (Non)-Convex Stochastic Optimization via Conditional Gradient and Gradient Updates. pages 3455–3464, 2018.

[5] Amrit Singh Bedi, Alec Koppel, and Ketan Rajawat. Nonparametric Compositional Stochastic Optimization. *arXiv preprint, arXiv:1902.06011*, February 2019.

[6] Sergio Bruno, Shabbir Ahmed, Alexander Shapiro, and Alexandre Street. Risk Neutral and Risk-Averse Approaches to Multistage Renewable Investment Planning under Uncertainty. *European Journal of Operational Research*, 250(3):979–989, May 2016. ISSN 03772217. doi: 10.1016/j.ejor.2015.10.013.

[7] Adrian Rivera Cardoso and Huan Xu. Risk-Averse Stochastic Convex Bandit. In *International Conference on Artificial Intelligence and Statistics*, volume 89, pages 39–47, April 2019.

[8] Yuntian Chen, Haibin Chang, Jin Meng, and Dongxiao Zhang. Ensemble Neural Networks (ENN): A gradient-free stochastic method. *Neural Networks*, 110:170–185, February 2019. ISSN 18792782. doi: 10.1016/j.neunet.2018.11.009.

[9] Zhiping Chen and Yi Wang. Two-Sided Coherent Risk Measures and Their Application in Realistic Portfolio Optimization. *Journal of Banking and Finance*, 32(12):2667–2673, 2008. ISSN 03784266. doi: 10.1016/j.jbankfin.2008.07.004.

[10] A. R. (Andrew R.) Conn, Katya. Scheinberg, and Luis N. Vicente. *Introduction to Derivative-Free Optimization*. Society for Industrial and Applied Mathematics/Mathematical Programming Society, 2009. ISBN 9780898716689.

[11] John C. Duchi, Michael I. Jordan, Martin J. Wainwright, and Andre Wibisono. Optimal Rates for Zero-Order Convex Optimization: The Power of Two Function Evaluations. *IEEE Transactions on Information Theory*, 61(5):2788–2806, May 2015. ISSN 00189448. doi: 10.1109/TIT.2015.2409256.

[12] Gerald B. Folland. *Real Analysis: Modern Techniques and their Applications.* John Wiley & Sons, 2nd edition, 1999.

[13] Tianwen Fu, Xinkai Zhuang, Yongchang Hui, and Jia Liu. Convex Risk Measures Based on Generalized Lower Deviation and Their Applications. *International Review of Financial Analysis*, 52:27–37, 2017. ISSN 10575219. doi: 10.1016/j.irfa.2017.04.008.

[14] Saeed Ghadimi and Guanghui Lan. Stochastic First- and Zeroth-order Methods for Nonconvex Stochastic Programming. *SIAM Journal on Optimization*, 23(4):2341–2368, January 2013. ISSN 1052-6234. doi: 10.1137/120880811.

[15] Saeed Ghadimi, Guanghui Lan, and Hongchao Zhang. Mini-Batch Stochastic Approximation Methods for Nonconvex Stochastic Composite Optimization. *Mathematical Programming*, 155 (1-2):267–305, January 2016. ISSN 14364646. doi: 10.1007/s10107-014-0846-1.

[16] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep learning.* MIT Press, 2016. ISBN 9780262035613.

[17] Jun-ya Gotoh and Stan Uryasev. Support Vector Machines Based on Convex Risk Functions and General Norms. *Annals of Operations Research*, 249(1-2):301–328, February 2017. ISSN 15729338. doi: 10.1007/s10479-016-2326-x.

[18] Davood Hajinezhad and Michael M. Zavlanos. Gradient-Free Multi-Agent Nonconvex Nonsmooth Optimization. In *Proceedings of the IEEE Conference on Decision and Control*, volume 2018-Decem, pages 4939–4944. IEEE, December 2019. ISBN 9781538613955. doi: 10.1109/CDC.2018.8619333.

[19] Trevor Hastie, Robert Tibshirani, and Jerome Friedman. *The Elements of Statistical Learning.* Springer Series in Statistics. Springer New York, New York, NY, 2009. ISBN 978-0-387-84857-0. doi: 10.1007/978-0-387-84858-7.

[20] Wenjie Huang and William B. Haskell. Risk-Aware Q-learning for Markov Decision Processes. In *2017 IEEE 56th Annual Conference on Decision and Control, CDC 2017*, volume 2018-Janua, pages 4928–4933. IEEE, December 2018. ISBN 9781509028733. doi: 10.1109/CDC.2017.8264388.

[21] Daniel R. Jiang and Warren B. Powell. Risk-Averse Approximate Dynamic Programming with Quantile-Based Risk Measures. *Mathematics of Operations Research*, 43(2):554–579, November 2018. ISSN 15265471. doi: 10.1287/moor.2017.0872.

[22] Dionysios S. Kalogerias and Warren B. Powell. Recursive Optimization of Convex Risk Measures: Mean-Semideviation Models. *arXiv preprint, arXiv:1804.00636*, April 2018.

[23] Sung-Kyun Kim, Rohan Thakker, and Ali-akbar Agha-mohammadi. Bi-directional Value Learning for Risk-aware Planning Under Uncertainty. *IEEE Robotics and Automation Letters*, 4(3): 2493–2500, July 2019. ISSN 2377-3766. doi: 10.1109/LRA.2019.2903259.

[24] Wann-Jiun Ma, Chanwook Oh, Yang Liu, Darinka Dentcheva, and Michael M. Zavlanos. Risk-Averse Access Point Selection in Wireless Communication Networks. *IEEE Transactions on Control of Network Systems*, 5870(c):1–1, 2018. ISSN 2325-5870. doi: 10.1109/TCNS.2018.2792309.

[25] Somayeh Moazeni, Warren B. Powell, and Amir H. Hajimiragha. Mean-Conditional Value-at-Risk Optimal Energy Storage Operation in the Presence of Transaction Costs. *IEEE Transactions on Power Systems*, 30(3):1222–1232, May 2015. ISSN 08858950. doi: 10.1109/TPWRS.2014.2341642.

[26] Somayeh Moazeni, Warren B. Powell, Boris Defourny, and Belgacem Bouzaiene-Ayari. Parallel nonstationary direct policy search for risk-averse stochastic optimization. *INFORMS Journal on Computing*, 29(2):332–349, May 2017. ISSN 15265528. doi: 10.1287/ijoc.2016.0733.

[27] A. S. Nemirovsky and D. B. Yudin. *Problem Complexity and Method Efficiency in Optimization*. John Wiley & Sons, New York, 1983.

[28] Yurii Nesterov and Vladimir Spokoiny. Random Gradient-Free Minimization of Convex Functions. *Foundations of Computational Mathematics*, 17(2):527–566, April 2017. ISSN 16153383. doi: 10.1007/s10208-015-9296-2.

[29] Matthew Norton, Alexander Mafusalov, and Stan Uryasev. Soft Margin Support Vector Classification as Buffered Probability Minimization. *Journal of Machine Learning Research*, 18(68): 1–43, 2017.

[30] Włodzimierz Ogryczak and Andrzej Ruszczyński. From Stochastic Dominance to Mean-Risk Models: Semideviations as Risk Measures. *European Journal of Operational Research*, 116(1): 33–50, July 1999. ISSN 03772217. doi: 10.1016/S0377-2217(98)00167-2.

[31] Włodzimierz Ogryczak and Andrzej Ruszczyński. Dual Stochastic Dominance and Related Mean-Risk Models. *SIAM Journal on Optimization*, 13(1):60–78, 2002. ISSN 1052-6234. doi: 10.1137/S1052623400375075.

[32] Arvind A. Pereira, Jonathan Binney, Geoffrey A. Hollinger, and Gaurav S. Sukhatme. Risk-Aware Path Planning for Autonomous Underwater Vehicles using Predictive Ocean Models. *Journal of Field Robotics*, 30(5):741–762, September 2013. ISSN 15564967. doi: 10.1002/rob.21472.

[33] R. Tyrrell Rockafellar, Stan Uryasev, and Michael Zabarankin. Generalized Deviations in Risk Analysis. *Finance and Stochastics*, 10(1):51–74, January 2006. ISSN 09492984. doi: 10.1007/s00780-005-0165-8.

[34] Tyrrell R. Rockafellar, Stanislav P. Uryasev, and Michael Zabarankin. Deviation Measures in Risk Analysis and Optimization. *SSRN Electronic Journal*, December 2003. ISSN 1556-5068. doi: 10.2139/ssrn.365640.

[35] Andrzej P. Ruszczyński. *Nonlinear Optimization*. Princeton University Press, 2006. ISBN 9780691119151.

[36] Amir Sani, Alessandro Lazaric, and Rémi Munos. Risk-Aversion in Multi-Armed Bandits. In *Advances in Neural Information Processing Systems 25 (NIPS 2012)*, pages 3275–3283, 2012.

[37] Danjue Shang, Victor Kuzmenko, and Stan Uryasev. Cash Flow Matching with Risks Controlled by Buffered Probability of Exceedance and Conditional Value-at-Risk. *Annals of Operations Research*, 260(1-2):501–514, January 2018. ISSN 15729338. doi: 10.1007/s10479-016-2354-6.

[38] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory*. Society for Industrial and Applied Mathematics, 2nd edition, 2014. ISBN 089871687X. doi: http://dx.doi.org/10.1137/1.9780898718751.

[39] James C. Spall. *Introduction to Stochastic Search and Optimization: Estimation, Simulation, and Control*. Wiley-Interscience, 2003. ISBN 0471441902.

[40] Aviv Tamar, Yinlam Chow, Mohammad Ghavamzadeh, and Shie Mannor. Sequential Decision Making with Coherent Risk. *IEEE Transactions on Automatic Control*, 62(7):3323–3338, July 2017. ISSN 00189286. doi: 10.1109/TAC.2016.2644871.

[41] Gavin Taylor, Ryan Burmeister, Zheng Xu, Bharat Singh, Ankit Patel, and Tom Goldstein. Training Neural Networks Without Gradients: A Scalable ADMM Approach. In *33rd International Conference on Machine Learning (ICML 2016)*, pages 2722–2731, June 2016.

[42] Vladimir N. Vapnik. *The Nature of Statistical Learning Theory*. Springer, 2000. ISBN 0387987800.

[43] Constantine Alexander Vitt, Darinka Dentcheva, and Hui Xiong. Risk-Averse Classification. *Annals of Operations Research*, August 2019. ISSN 0254-5330. doi: 10.1007/s10479-019-03344-6.

[44] Mengdi Wang, Ethan X. Fang, and Han Liu. Stochastic Compositional Gradient Descent: Algorithms for Minimizing Compositions of Expected-Value Functions. *Mathematical Programming*, 161(1-2):419–449, January 2017. ISSN 14364646. doi: 10.1007/s10107-016-1017-3.

[45] Shuoguang Yang, Mengdi Wang, and Ethan X. Fang. Multilevel Stochastic Gradient Methods for Nested Composition Optimization. *SIAM Journal on Optimization*, 29(1):616–659, January 2019. ISSN 1052-6234. doi: 10.1137/18M1164846.

[46] Pengqian Yu, William B. Haskell, and Huan Xu. Approximate Value Iteration for Risk-Aware Markov Decision Processes. *IEEE Transactions on Automatic Control*, 63(9):3135–3142, 2018. ISSN 15582523. doi: 10.1109/TAC.2018.2790261.

[47] Deming Yuan and Daniel W.C. Ho. Randomized Gradient-Free Method for Multiagent Optimization over Time-Varying Networks. *IEEE Transactions on Neural Networks and Learning Systems*, 26(6):1342–1347, June 2015. ISSN 21622388. doi: 10.1109/TNNLS.2014.2336806.

[48] Lifeng Zhou and Pratap Tokekar. An Approximation Algorithm for Risk-averse Submodular Optimization. *arXiv preprint, arXiv:1807.09358*, July 2018.