

Predictive Learning on Sign-Valued Hidden Markov Trees

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Abstract

We provide high-probability sample complexity guarantees for exact structure recovery and accurate Predictive Learning using noise-corrupted samples from an acyclic (tree-shaped) graphical model. The hidden variables follow a tree-structured Ising model distribution, whereas the observable variables are generated by a binary symmetric channel, taking the hidden variables as its input. This model arises naturally in a variety of applications, such as in physics, biology, computer science, and finance. The noiseless structure learning problem has been studied earlier by Bresler and Karzand (2018); this paper quantifies how noise in the hidden model impacts the sample complexity of structure learning and predictive distributional inference by proving upper and lower bounds on the sample complexity. Quite remarkably, for any tree with p vertices and probability of incorrect recovery $\delta > 0$, the order of necessary number of samples remains logarithmic as in the noiseless case, i.e., $\mathcal{O}(\log(p/\delta))$, for both aforementioned tasks. We also present a new equivalent of Isserlis' Theorem for sign-valued tree-structured distributions, yielding a new low-complexity algorithm for higher order moment estimation.

1. Introduction

Graphical models are a useful tool for modeling high-dimensional structured data. Indeed, the graph captures structural dependencies: its edge set corresponds to (often physical) interactions between variables. There is a long and deep literature on graphical models (see Koller and Friedman (2009) for a comprehensive introduction), and they have found wide applications in areas such as image processing and vision (Liu et al., 2017; Schwing and Urtasun, 2015; Lin et al., 2016a; Morningstar and Melko, 2017; Wu et al., 2017; Li and Wand, 2016), artificial intelligence more broadly (Wang et al., 2017; Wainwright et al., 2003), signal processing (Wisdom et al., 2016; Kim and Smaragdis, 2013), and gene regulatory networks (Zuo et al., 2017; Banf and Rhee, 2017), to name a few.

An *undirected graphical model*, or *Markov random field* (MRF) in particular, is defined in terms of a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which models the Markov properties of a joint distribution on variables $\mathbf{X} = (X_1, X_2, \dots, X_p)$ where $p = |\mathcal{V}|$. A *tree-structured graphical model* is one in which \mathcal{G} is a tree. We denote the tree-structured model as $T = (\mathcal{V}, \mathcal{E})$. In this paper, we consider binary models on $2p$ variables (\mathbf{X}, \mathbf{Y}) , where the joint distribution $p(\cdot)$ of \mathbf{X} is a tree-structured Ising model distribution on $\{-1, +1\}^p$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_p)$ is a noisy version of \mathbf{X} , where $Y_i = N_i X_i$ and $\{N_i\}$ are independent and identically distributed (i.i.d.) Rademacher noise. We refer to \mathbf{X} as the *hidden layer* and \mathbf{Y} as the *observed layer*. Under this setting, our objective is to recover the underlying tree structure of the hidden layer \mathbf{X} (with high probability) using only the noisy observations \mathbf{Y} . This is non-trivial because \mathbf{Y} does *not* itself follow any tree structure; this is similar to more traditional problems in nonlinear filtering, where a Markov process of known distribution (and thus, of known structure) is observed through noisy measurements (Arulampalam et al., 2002; Jazwinski, 2007; Van Handel, 2009; Douc et al., 2011; Kalogerias and Petropulu, 2016). The sample complexity of the noiseless version of our model was recently studied by Bresler and Karzand (2018), where the well-known Chow-Liu algorithm (Chow and Liu, 1968) is employed for tree reconstruction. Like them, we also consider the Chow-Liu algorithm (Chow and Liu, 1968) (more precisely, a slightly modified version of it) in this paper, as well.

1.1 Statement of Contributions

We are interested in answering the following general question: *how does noise affect the sample complexity of the structure learning procedure?* That is, given *only* noisy observations, our goal is to learn the tree structure of the hidden layer, in a well-defined and meaningful sense. In turn, the estimated structure is an essential statistic for estimating the underlying distribution of the hidden layer, allowing for Predictive Learning.

Specifically, based on the structure estimate, we are also interested in appropriately approximating the tree-structured distribution under study, which can then be used for accurate predictions, in regard to the statistical structure of the underlying tree. We also consider the problem of hidden layer higher order moment estimation of sign-valued hidden Markov fields on trees and, in particular, how such estimation can be efficiently performed, on the basis of noisy observations.

Our contributions may be summarized as follows:

- A lower bound on the number of samples needed to recover the exact hidden structure with high probability, by using the Chow-Liu algorithm. Complementary, an upper bound on the necessary number of samples for the hidden structure learning task.
- Determination of the sufficient and necessary number of samples for accurate predictive learning. We analyze the sample complexity of learning distribution estimates, which can accurately provide predictions on the hidden tree. The estimates are computed through noisy data.
- A closed-form expression *and* a computationally efficient estimator for higher-order moment estimation in sign-valued tree-structured Markov random fields.
- Sample complexity analysis for accurate distribution estimates with respect to the KL-divergence.

1.2 Structure Learning for Undirected Graphical Models and Related Work

For a detailed review of methods involving undirected and directed graphical models, see the relevant article by Drton and Maathuis (2017). In general, learning the structure of a graphical model from samples can be intractable (Karger and Srebro, 2001; Højsgaard et al., 2012). For general graphs, neighborhood selection methods (Bresler, 2015; Ray et al., 2015; Jalali et al., 2011) estimate the conditional distribution for each vertex, to learn the neighborhood of each node and therefore the full structure. These approaches may use greedy search or ℓ_1 regularization. For Gaussian or Ising models, ℓ_1 -regularization (Ravikumar et al., 2010), the GLasso (Yuan and Lin, 2007; Banerjee et al., 2008), or coordinate descent approaches (Friedman et al., 2008) have been proposed, focusing on estimating the non-zero entries of the precision (or interaction) matrix. Model selection can also be performed using score matching methods (Hyvärinen, 2005, 2007; Nandy et al., 2015; Lin et al., 2016b), or Bayesian information criterion methods (Foygel and Drton, 2010; Gao et al., 2012; Barber et al., 2015). Other works address non-Gaussian models such as elliptical distributions, t-distribution models or latent Gaussian data (Vogel and Fried, 2011; Vogel and Tyler, 2014; Bilodeau, 2014; Finegold and Drton, 2011), or even mixed data (Fan et al., 2017).

For tree- or forest-structured models, exact inference and the structure learning problem are significantly simpler: the Chow-Liu algorithm provides an estimate of the tree or forest structure of the underlying graph (Bresler and Karzand, 2018; Chow and Liu, 1968; Wainwright et al., 2008; Tan et al., 2011; Liu et al., 2011; Edwards et al., 2010; Daskalakis et al., 2018). Furthermore, marginal distributions and maximum values are simpler to compute using a variety of algorithms (sum-product, max-product, message passing, variational inference) (Lauritzen, 1996; Pearl, 1988; Wainwright et al., 2008, 2003)).

The noiseless counterpart of the model considered in this paper was studied recently by Bresler and Karzand (2018); in this paper, we extend their results to the hidden case, where samples from a tree-structured Ising model are passed through a binary symmetric channel with crossover probability $q \in [0, 1/2]$. Of course, in the special case of a linear graph, our model reduces to a hidden Markov model. Latent variable models are often considered in the literature when some variables of the graph are deterministically unobserved (Chandrasekaran et al., 2010; Anandkumar and Valluvan, 2013; Ma et al., 2013; Anandkumar et al., 2014). Our model is most similar to that studied by Chaganty et al. (Chaganty and Liang, 2014), in which a hidden model is considered with a discrete exponential distribution and Gaussian noise. They solve the parameter estimation problem by using moment matching and pseudo-likelihood methods; the structure can be recovered indirectly using the estimated parameters.

Notation. Boldface indicates a vector or tuple and calligraphic face for sets and trees. The sets of even and odd natural numbers are $2\mathbb{N}$ and $2\mathbb{N} + 1$ respectively. For an integer n , define $[n] \triangleq \{1, 2, \dots, n\}$. The indicator function of a set A is $\mathbf{1}_A$. For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = [p]$ indexes the set of variables $\{X_1, X_2, \dots, X_p\}$, for any pair of vertices $i, j \in \mathcal{V}$ the correlation $\mu_{ij} \triangleq \mathbb{E}[X_i X_j]$ and for any edge $e = (i, j) \in \mathcal{E}$ it is $\mu_e \triangleq \mathbb{E}[X_i X_j]$. For two nodes w, \tilde{w} of a tree, the term $\text{path}(w, \tilde{w})$ denotes the set of edges in the unique path with endpoints w and \tilde{w} . The binary symmetric channel $\text{BSC}(q)^p$ is a conditional distribution from $\{-1, 1\}^p \rightarrow \{-1, 1\}^p$ that acts componentwise independently on \mathbf{X} to generate \mathbf{Y} , where $X_i = N_i Y_i$ and \mathbf{N} is a vector of i.i.d. Rademacher variables equal to

Symbol	Meaning
p	number of variables, number of nodes in the tree
α	minimum $ \theta_{ij} $ in the Ising model
β	maximum $ \theta_{ij} $ in the Ising model
$\mathcal{P}_T(\alpha, \beta)$	set of tree-structured Ising models with $\alpha < \theta_{ij} < \beta$
n	number of samples
q	crossover probability of the BSC, $q \in [0, \frac{1}{2}]$
c	$1 - 2q$
\mathbf{D}_{KL}	KL divergence
\mathbf{S}_{KL}	symmetric KL divergence
$I(X, Y)$	mutual information of X, Y
$\mathcal{L}^{(2)}$	ssTV defined in (14)
T^{CL}	Chow-Liu-estimated structure for the noiseless model
$\text{T}^{\text{CL}}_{\dagger}$	Chow-Liu-estimated structure for the hidden model
$\text{path}_T(w, w')$	The set of edges which connects the nodes $w, w' \in \mathcal{V}_T$
Π_T	Reverse-I projection (22)
$\hat{\mu}_{i,j}$	$\frac{1}{n} \sum_{k=1}^n (X_i X_j)^{(k)}$
$\hat{\mu}_{i,j}^{\dagger}$	$\frac{1}{n_{\dagger}} \sum_{k=1}^{n_{\dagger}} (Y_i Y_j)^{(k)}$

Table 1: Notation/Definitions.

+1 with probability $1 - q$. We use the symbol \dagger to indicate the corresponding quantity for the observable (noisy) layer. For instance, $p_{\dagger}(\cdot)$ is the probability mass function of \mathbf{Y} and $\mu_{i,j}^{\dagger} \triangleq \mathbb{E}[Y_i Y_j]$ corresponds to the correlation of variables Y_i, Y_j , where Y_i generates noisy observations of X_i , for any $i \in \mathcal{V}$. Also, $\text{BSC}(q)^p$ denotes a binary symmetric channel with crossover probability q and blocklength p . For our readers' convenience, we summarize the notation in Table 1.

2. Preliminaries and Problem Statement

In this section, we introduce our model of *hidden sign-valued Markov random fields on trees*.

2.1 Undirected Graphical Models

We consider *sign-valued graphical models* where the joint distribution $p(\cdot)$ has support $\{-1, +1\}^p$. Let $\mathbf{X} = (X_1, X_2, \dots, X_p) \in \{-1, +1\}^p$ be a collection of sign-valued (binary) random variables. Then, $\mathbf{1}_{X_i=x_i} \equiv (1 + x_i X_i)/2$. We consider distributions on \mathbf{X} of the form

$$p(\mathbf{x}) = \mathbb{E} \left[\prod_{i=1}^p \mathbf{1}_{X_i=x_i} \right] = \frac{1}{2^p} \left[1 + \sum_{k \in [p]} \sum_{\mathcal{S} \subset \mathcal{V}: |\mathcal{S}|=k} \mathbb{E} \left[\prod_{s \in \mathcal{S}} X_s \right] \prod_{s \in \mathcal{S}} x_s \right], \quad \mathbf{x} \in \{-1, +1\}^p. \quad (1)$$

In this paper we assume that the marginal distributions of the X_i are uniform, that is,

$$\mathbb{P}(X_i = \pm 1) = \frac{1}{2}, \quad \forall i \in \mathcal{V}. \quad (2)$$

Thus, $\mathbb{E}[X_i] = 0$, for all $i \in \mathcal{V}$. A distribution is Markov with respect to a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if for every node i in the set \mathcal{V} it is true that $\mathbb{P}(X_i | \mathbf{x}_{\mathcal{V} \setminus \{i\}}) = \mathbb{P}(X_i | \mathbf{x}_{\mathcal{N}(i)})$, where $\mathcal{N}(i)$ is the set of neighbors of i in \mathcal{G} . One subclass of distributions for which the Markov property holds is the *Ising model*, in which each random variable X_i is sign-valued and the hypergraph is a simple undirected graph, indicating that variables have only pairwise and unitary interactions. The joint distribution for the *Ising model* with zero external field is given by

$$p(\mathbf{x}) = \frac{1}{Z(\theta)} \exp \left\{ \sum_{(s,t) \in \mathcal{E}} \theta_{st} x_s x_t \right\}, \quad \mathbf{x} \in \{-1, 1\}^p. \quad (3)$$

$\{\theta_{st} : (s, t) \in \mathcal{E}\}$ are parameters of the model representing the interaction strength of the variables and $Z(\cdot) \in (0, \infty)$ is the *partition function*. These interactions are expressed through potential functions $\exp(\theta_{st} x_s x_t)$ which ensure that the Markov property holds with respect to the graph $G = (\mathcal{V}, \mathcal{E})$. Next, we discuss the properties of distributions of the form of (1), *which are Markov with respect to a tree*.

2.2 Sign-Valued Markov Fields on Trees

From prior work by Lauritzen (1996), it is known that any distribution $p(\cdot)$ which is Markov with respect to a tree (or forest) $T = (\mathcal{V}, \mathcal{E})$ factorizes as

$$p(\mathbf{x}) = \prod_{i \in \mathcal{V}} p(x_i) \prod_{(i,j) \in \mathcal{E}} \frac{p(x_i, x_j)}{p(x_i)p(x_j)}, \quad \mathbf{x} \in \{-1, +1\}^p, \quad (4)$$

and we call $p(\cdot)$ as tree (forest) structured distribution, to indicate the factorization property. If the distribution $p(\cdot)$ has the form of (1) with $\mathbb{P}(X_i = \pm 1) = 1/2$, for all $i \in \mathcal{V}$, and is Markov with respect to a tree T , then

$$p(\mathbf{x}) = \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2} \quad (5)$$

and

$$\mathbb{E}[X_i X_j] = \prod_{e \in \text{path}(i,j)} \mu_e, \quad \text{for all } i, j \in \mathcal{V}. \quad (6)$$

(see Appendix A, Lemma A.1). Additionally, let us state the definition of the so-called *Correlation (coefficient) Decay Property (CDP)*, which will be of central importance in our analysis.

Definition 1 *The CDP holds if and only if $|\mathbb{E}[X_i X_k]| \geq |\mathbb{E}[X_\ell X_m]|$ for all tuples $\{i, k, \ell, m\} \subset \mathcal{V}$ such that $\text{path}(i, k) \subset \text{path}(\ell, m)$.*

The CDP is a well known attribute of acyclic Markov fields (see, e.g., Tan et al. (2010), Bresler and Karzand (2018)). Further, it is true that the products $X_i X_j$ for all $(i, j) \in \mathcal{E}$ are independent and the CDP holds for every $p(\cdot)$ of the form of (1), which factorizes with

respect to a tree (see Lemma A.2, Appendix A). This is a consequence of property (6) and the inequality $|\mu_e| \leq 1$, for all $e \in \mathcal{E}$. We can interpret the CDP as a type of data processing inequality (see Cover and Thomas (2012)). The connection is clear through the relationship between the mutual information $I(X_i, X_j)$ and the correlations $\mathbb{E}[X_i X_j]$, namely,

$$I(X_i, X_j) = \frac{1}{2} \log_2 \left((1 - \mathbb{E}[X_i X_j])^{1 - \mathbb{E}[X_i X_j]} (1 + \mathbb{E}[X_i X_j])^{1 + \mathbb{E}[X_i X_j]} \right), \quad (7)$$

for any pair of nodes $i, j \in \mathcal{V}$. This expression shows that the mutual information is a symmetric function of $\mathbb{E}[X_i X_j]$ and increasing with respect to $|\mathbb{E}[X_i X_j]|$ (see also Lemma A.5, Appendix A).

Tree-structured Ising models: Despite its simple form, the Ising model has numerous useful properties. In particular, (5), (6) hold for any tree-structured Ising model with uniform marginal distributions and $\theta_r = 0$ for all $r \in \mathcal{V}$. Furthermore,

$$\mathbb{E}[X_i X_j] = \tanh \theta_{ij}, \quad \forall (i, j) \in \mathcal{E}_T, \quad (8)$$

which implies that

$$p(\mathbf{x}) = \frac{1}{2} \prod_{(i,j) \in \mathcal{E}_T} \frac{1 + x_i x_j \tanh \theta_{ij}}{2}, \quad \mathbf{x} \in \{-1, 1\}^p, \quad \alpha < |\theta_{ij}| \leq \beta, \quad (9)$$

$$\mathbb{E}[X_i X_j] = \prod_{e \in \text{path}(i,j)} \mu_e = \prod_{e \in \text{path}(i,j)} \tanh(\theta_e), \quad \forall i, j \in \mathcal{V}. \quad (10)$$

A short argument showing (8) and (9) is included in Appendix A, Lemma A.3. For the rest of the paper, we assume a tree-structured Ising model for the hidden variable \mathbf{X} , that is, the distribution of \mathbf{X} has the form of (5). We also impose a reasonable compactness assumption on the respective interaction parameters, as follows.

Assumption 1 *There exist α and β such that for the distribution $p(\cdot)$, $0 < \alpha \leq |\theta_{st}| \leq \beta < \infty$ for all $(s, t) \in \mathcal{E}$.*

For a fixed tree structure T , and for future reference, we hereafter let $\mathcal{P}_T(\alpha, \beta)$ be the class of Ising models satisfying Assumption 1.

2.3 Hidden Sign-Valued Tree-Structured Models

The problem considered in this paper is that of learning a tree-structured model from corrupted observations. Instead of observing samples $\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(n)$, we observe $\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(n)$ where $\mathbf{y}(i)$ is a noisy version of $\mathbf{x}(i)$ for $i \in [n]$. To formalize this, consider a hidden Markov random field whose hidden layer \mathbf{X} is an Ising model with respect to a tree, i.e., $\mathbf{X} \sim p(\cdot) \in \mathcal{P}_T(\alpha, \beta)$, as defined in (9). The observed variables \mathbf{Y} are formed by setting $Y_r = N_r X_r$ for all $r \in \mathcal{V}$, where $\{N_r\}$ are i.i.d. Rademacher(q) random variables. Let $p_{\dagger}(\cdot)$ be the distribution of the observed variables \mathbf{Y} . We can think of \mathbf{Y} as the result of passing \mathbf{X} through a binary symmetric channel BSC(q) ^{p} . This yields the following formula for $p_{\dagger}(\cdot)$ as a function of higher-order moments:

$$\mathbb{E}[N_r] = 1 - 2q, \quad \forall r \in \mathcal{V}, \quad \text{and } q \in \left[0, \frac{1}{2}\right], \quad (11)$$

Algorithm 1 Chow – Liu

Require: $\mathcal{D} = \{\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(n_{\dagger})\}$, where $\mathbf{y}(k)$ is the k^{th} observation of \mathbf{Y}

- 1: Compute $\hat{\mu}_{i,j}^{\dagger} \leftarrow \frac{1}{n_{\dagger}} \sum_{k=1}^{n_{\dagger}} y_i(k)y_j(k)$, for all $i, j \in \mathcal{V}$
 - 2: *For* **Chow – Liu – CC** :
 - 3: **if** $\hat{\mu}_{i,j}^{\dagger} > (1 - 2q)^2 \tanh(\beta)$ **then**
 - 4: $\hat{\mu}_{i,j}^{\dagger} \leftarrow (1 - 2q)^2 \tanh(\beta)$
 - 5: **else if** $\hat{\mu}_{i,j}^{\dagger} < -(1 - 2q)^2 \tanh(\beta)$ **then**
 - 6: $\hat{\mu}_{i,j}^{\dagger} \leftarrow -(1 - 2q)^2 \tanh(\beta)$
 - 7: **return** $T_{\dagger}^{\text{CL}} \leftarrow \text{MaximumSpanningTree}\left(\cup_{i \neq j} \left\{ \left| \hat{\mu}_{i,j}^{\dagger} \right| \right\}\right)$
-

$$\mu_{r,s}^{\dagger} \triangleq \mathbb{E}[Y_r Y_s] = \mathbb{E}[N_r X_r N_s X_s] = (1 - 2q)^2 \mathbb{E}[X_r X_s], \quad \forall r, s \in \mathcal{V}. \quad (12)$$

The constant $c \triangleq 1 - 2q$ will feature prominently in the analysis. The distribution $p_{\dagger}(\cdot)$ of \mathbf{Y} also has support $\{-1, +1\}^p$, and so the joint distribution satisfies the general form (1). Since the marginal distribution of each Y_r is also uniform, $\mathbb{E}[Y_r] = 0$ for all $r \in \mathcal{V}$, (1) and (11) yield

$$p_{\dagger}(\mathbf{y}) = \mathbb{E}\left[\prod_{i=1}^p \mathbf{1}_{Y_i=y_i}\right] = \frac{1}{2^p} \left[1 + \sum_{k \in [p] \cap 2\mathbb{N}} c^k \sum_{\mathcal{S} \subset \mathcal{V}: |\mathcal{S}|=k} \mathbb{E}\left[\prod_{s \in \mathcal{S}} X_s\right] \prod_{s \in \mathcal{S}} y_s \right], \quad \mathbf{y} \in \{-1, 1\}^p. \quad (13)$$

The moments of the hidden variables $\mathbb{E}\left[\prod_{s \in \mathcal{S}} X_s\right]$ in (13) can be expressed as products of the pairwise correlations $\mathbb{E}[X_s X_t]$, for any $(s, t) \in \mathcal{E}_T$ (Section 3.3, Theorem 3.6). From (13) it is clear that the distribution $p_{\dagger}(\cdot)$ of \mathbf{Y} does not factorize with respect to any tree, that is, $p_{\dagger}(\cdot) \notin \mathcal{P}_T(\alpha, \beta)$ in general.¹ The next subsections present the contribution and the most important questions which are handled in this paper.

2.4 Hidden Structure Estimation

We are interested in characterizing the *sample complexity* of structure recovery: given data generated from $p(\cdot) \in \mathcal{P}_T(\alpha, \beta)$ for an unknown tree T , what is the minimum number n_{\dagger} of samples $\{\mathbf{y}^{(i)}, i \in [n_{\dagger}]\}$ needed to recover the (unweighted) edge set of T with high probability? In particular, we would like to quantify how n_{\dagger} depends on the crossover probability q . Intuitively, noise makes "weak" edges to appear "weaker", and the sample complexity is expected to be an increasing function of q . Because the distribution $p_{\dagger}(\cdot)$ of the observable variables does not follow the tree structure T , this problem does not follow directly from the noiseless case. In this work, we use the Chow-Liu algorithm to estimate the hidden tree structure. Specifically, we analyze the sample complexity of Algorithm 1. Our model allows us to retrieve a coherent variation of the original algorithm by Chow and Liu (1968). The consistency of Algorithm 1 is explained in depth in the next Sections (see Section 3.1, Section 4.1)

1. Lemma F.3 shows the structure preserving property for the observable layer for specific choices of the hidden layer's tree structure.

2.5 Evaluating the Accuracy of the Estimated Distribution

In addition to recovering the graph structure, we are interested in the "goodness of fit" of the estimated distribution. We measure this through the "small set Total Variation" (or "ssTV") distance as defined by Bresler and Karzand (2018):

$$\mathcal{L}^{(k)}(P, Q) \triangleq \sup_{\mathcal{S}:|\mathcal{S}|=k} d_{\text{TV}}(P_{\mathcal{S}}, Q_{\mathcal{S}}), \quad (14)$$

where $P_{\mathcal{S}}, Q_{\mathcal{S}}$ are the marginal distributions of P, Q on the set $\mathcal{S} \subset \mathcal{V}$, d_{TV} is the total variation distance, and $k = 2$. If Q is an estimate of P , the norm $\mathcal{L}^{(k)}$ guarantees predictive accuracy because (Bresler and Karzand, 2018, Section 3.2)

$$\mathbb{E}_{X_{\mathcal{S}}} \left[|P(X_i = +1|X_{\mathcal{S}}) - Q(X_i = +1|X_{\mathcal{S}})| \right] \leq 2\mathcal{L}^{(|\mathcal{S}|+1)}(P, Q). \quad (15)$$

We propose an estimator for the distribution of the hidden variables, $p(\cdot) \in \mathcal{P}_{\text{T}}(\alpha, \beta)$, given only noisy observations. We also design the estimator to factorize according to the estimated structure $\text{T}_{\dagger}^{\text{CL}}$, where the latter results as the output of the *clipped Chow-Liu*, or *Chow-Liu-CC algorithm* (see Algorithm 1). The Chow-Liu-CC algorithm constitutes a slightly modified version of the original Chow-Liu algorithm. In particular, for some k , $|\hat{\mu}_k^{\dagger}|$ is greater than $(1 - 2q)^2 \tanh(\beta)$, then its value is set to $(1 - 2q)^2 \tanh(\beta)$. This extra step reduces the error of the estimation and simplifies the analysis (see Section 4.4). Our main result gives a lower bound on the number of samples needed to guarantee accurate estimation (in the sense of small "ssTV"), with high probability.

3. Main Results

The main question asked by this paper is as follows: *What is the impact of noise on the sample complexity of learning a tree-structured graphical model in order to make predictions?* This corresponds to sampling variables \mathbf{Y} generated by sampling \mathbf{X} from the model (3) and randomly flipping each sign independently with probability q (*Binary symmetric channel*). The Chow-Liu algorithm estimates the hidden structure through corrupted by noise observations, which consist the output of the BSC. Hidden (binary) model structures arise in a variety of applications. A typical example is classification where a subset of the data is misclassified. In such a case, corrupted data are observed, however, we are still able to retrieve the underlying structure by considering the appropriate number of samples.

Regarding this work, we firstly derive the Chow-Liu algorithm's sufficient and necessary number of samples bounds for exact hidden structure recovery, assuming that only noisy observations are available. Theorem 3.1 and Theorem 3.2 correspond to the sufficient and necessary samples result respectively.

Secondly, we use the structure statistic to derive an accurate estimate of the hidden layer's probability distribution. The distribution estimate is computed to be accurate under the "small set Total Variation" or "ssTV" utility measure, which was introduced by Bresler and Karzand (2018). Furthermore, the estimator of the distribution factorizes with respect to the structure estimate, while the "ssTV" metric ensures that the estimated distribution is a trustworthy predictor. Theorem 3.3 and Theorem 3.4 give the sufficient and necessary sample complexity for accurate distribution estimation from noisy samples. These theorems

generalize the results for the noiseless case ($q = 0$) by Bresler and Karzand (2018) and lead to interesting connections between structure learning on hidden models and data processing inequalities.

The third part of the results includes Theorem 3.6, which gives the closed form expression for higher order moments of sign-valued Markov fields on trees is an equivalent of Isserlis' theorem. Based on Theorem 3.6, we propose a low complexity algorithm, which estimates any higher order moment of the hidden variables, while it requires as input the estimated tree structure and estimates of the pairwise correlations (both evaluated from corrupted by noise observations).

Finally, Theorem 3.7 gives the sufficient number of samples for distribution estimation, when the symmetric KL divergence is considered as utility measure. The last gives rise to extensions of testing algorithms Daskalakis et al. (2018) under a hidden model setting.

3.1 Tree Structure Learning from Noisy Observations

Our goal is to learn the tree structure T of an Ising model with parameters $|\theta_{st}| \in [\alpha, \beta]$, when the nodes X_i are hidden variables and we observe $Y_i \triangleq N_i X_i$, $i \in \mathcal{V}$, where $N_i \sim \text{Rademacher}(q)$ are i.i.d, for all $i \in \mathcal{V}$ and for all $q \in [0, 1/2)$. We derive the estimated structure $\mathsf{T}_\dagger^{\text{CL}}$ by applying the Chow-Liu algorithm (Algorithm 1) (Chow and Liu, 1968).

Instead of mutual information estimates, the Chow-Liu algorithm (Algorithm 1) requires correlations estimates, which are sufficient statistics because of expression (7). Further, it can consistently recover the hidden structure through noisy observations. The last is true because of the *order preserving* property of the mutual information. That is, the stochastic mapping $\mathbf{X} \xrightarrow{\text{BSC}(q)^p} \mathbf{Y}$ allows structure recovery of \mathbf{X} by observing \mathbf{Y} , because for any tuple $X_i, X_j, X_{i'}, X_{j'}$ such that $I(X_i; X_j) \leq I(X_{i'}, X_{j'})$, it is true that $I(Y_i; Y_j) \leq I(Y_{i'}, Y_{j'})$. The proof directly comes from (7) and (12). In addition, the monotonicity of mutual information with respect to the absolute values of correlations allows us to apply the Chow-Liu algorithm directly on the estimated correlations $\hat{\mu}_{i,j}^\dagger \triangleq 1/n_\dagger \sum_{k=1}^{n_\dagger} (Y_i)^{(k)} (Y_j)^{(k)}$. Notice that $\hat{\mu}_{i,j}^\dagger$ can be used as an alternative of $\hat{\mu}_{i,j}$, because of (12). The algorithm returns the maximum spanning tree $\mathsf{T}_\dagger^{\text{CL}}$. Further discussion about the Chow-Liu algorithm is given in Section 4.1. The following theorem provides the sufficient number of samples for exact structure recovery through noisy observations.

Theorem 3.1 (Sufficient number of samples for structure learning) *Let \mathbf{Y} be the output of a $\text{BSC}(q)^p$, with input variable $\mathbf{X} \sim \mathsf{p}(\cdot) \in \mathcal{P}_{\mathsf{T}}(\alpha, \beta)$. Fix a number $\delta \in (0, 1)$. If the number of samples n_\dagger of \mathbf{Y} satisfies the inequality*

$$n_\dagger \geq \frac{32 \left[1 - (1 - 2q)^4 \tanh \beta \right]}{(1 - 2q)^4 (1 - \tanh \beta)^2 \tanh^2 \alpha} \log \frac{2p^2}{\delta}, \quad (16)$$

then Algorithm 1 returns $\mathsf{T}_\dagger^{\text{CL}} = \mathsf{T}$ with probability at least $1 - \delta$.

Complementary to Theorem 3.1, our next result characterizes the necessary number of samples required for exact structure recovery. Specifically, we prove an upper on the sample complexity, which characterizes the necessary number of samples for any estimator ψ .

Theorem 3.2 (Necessary number of samples for structure learning) *Let \mathbf{Y} be the output of a $\text{BSC}(q)^p$, with input variable $\mathbf{X} \sim p(\cdot) \in \mathcal{P}_{\mathbb{T}}(\alpha, \beta)$. If the given number of samples of \mathbf{Y} satisfies the inequality*

$$n_{\dagger} < \frac{[1 - (4q(1 - q))^p]^{-1}}{16\alpha \tanh(\alpha)} e^{2\beta} \log(p), \quad (17)$$

then for any estimator ψ , it is true that

$$\inf_{\psi} \sup_{\substack{\mathbb{T} \in \mathcal{T} \\ p(\cdot) \in \mathcal{P}_{\mathbb{T}}(\alpha, \beta)}} \mathbb{P}(\psi(\mathbf{Y}_{1:n_{\dagger}}) \neq \mathbb{T}) > \frac{1}{2}. \quad (18)$$

It can be shown that the right hand-side of (16) is greater than the right-hand side of (17) for any q in $[0, 1/2)$ (and for all possible values of p, β, α), by simply comparing the two terms. Theorems 3.1 and 3.2 reduce to the noiseless setting by setting $q = 0$ (Bresler and Karzand (2018)). The sample complexity is increasing with respect to q , and structure learning is always feasible as long as $q \neq 1/2$. That is, to have the same probability of exact recovery we always need $n_{\dagger} \geq n$ since

$$\frac{[1 - (1 - 2q)^4 \tanh(\beta)]}{[(1 - 2q)^4 (1 - \tanh(\beta))]} \geq 1, \quad \forall q \in \left[0, \frac{1}{2}\right) \text{ and } \beta \in \mathbb{R}, \quad (19)$$

where n is the required samples under a noiseless setting assumption. Furthermore,

$$\frac{1}{1 - (4q(1 - q))^p} \geq 1, \quad \forall q \in \left[0, \frac{1}{2}\right) \text{ and } p \in \mathbb{N}, \quad (20)$$

which shows that the sample complexity in a hidden model is greater than the noiseless case ($q = 0$), for any measurable estimator (Theorem 3.2). When q approaches $1/2$, the sample complexity goes to infinity, $n_{\dagger} \rightarrow \infty$, which makes structure learning impossible. Theorem 3.2 is a non-trivial extension of Theorem 3.1 by Bresler and Karzand (2018) to our hidden model. Our results combines Bresler's and Karzand's method and a strong data processing inequality by Polyanskiy and Wu (2017, Evaluation of the BSC). Upper bounds on the symmetric KL divergence for the output distribution $p_{\dagger}(\cdot)$ can not be found in a closed form. However, by using the SDPI, we manage to capture the dependence of the bound on the parameters α, β, q and derive a non-trivial result. When p goes to ∞ , the bound becomes trivial since: $\lim_{p \rightarrow \infty} 1/[1 - (4q(1 - q))^p] \rightarrow 1$, which gives the classical data processing inequality (contraction of KL divergence for finite alphabets, Raginsky (2016); Polyanskiy and Wu (2017)).

3.2 Predictive Learning from Noisy Observations

In addition to recovering the structure of the hidden Ising model, we are interested in estimating the distribution $p(\cdot) \in \mathcal{P}_{\mathbb{T}}(\alpha, \beta)$ itself. If the $\mathcal{L}^{(2)}$ distance between the estimator and the true distribution is sufficiently small, then the estimated distribution is appropriate for predictive learning because of (15). For consistency, this distribution should factorize according to the structure estimate $\mathbb{T}_{\dagger}^{\text{CL}}$ and for the predictive learning part, the estimate

T_{\dagger}^{CL} is considered the output of the Chow-Liu-CC algorithm (see Algorithm 1). We continue by defining the distribution estimator of $p(\cdot)$ as

$$\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \triangleq \frac{1}{2} \prod_{(i,j) \in \mathcal{E}_{T_{\dagger}^{\text{CL}}}} \frac{1 + x_i x_j \frac{\hat{\mu}_{i,j}^{\dagger}}{(1-2q)^2}}{2}. \quad (21)$$

The estimator (21) can be defined for any $q \in [0, 1/2)$. For $q = 0$ it reduces to that in the noiseless case, since $T_{\dagger}^{\text{CL}} \equiv T^{\text{CL}}$, $\hat{\mu}_{i,j}^{\dagger} \equiv \hat{\mu}_{i,j}$, and thus $\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \equiv \Pi_{T^{\text{CL}}}(\hat{P})$. It is also closely related to the reverse information projection onto the tree-structured T Ising models (Bresler and Karzand, 2018), in the sense that

$$\Pi_T(P) = \underset{Q \in \mathcal{P}_T(\alpha, \beta)}{\text{argmin}} \mathbf{D}_{\text{KL}}(P||Q), \quad P \in \mathcal{P}_T(\alpha, \beta). \quad (22)$$

To compute $\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger})$, two sufficient statistics are required: the structure T_{\dagger}^{CL} and the set of second order moments (Bresler and Karzand, 2018; Chow and Liu, 1968), while q is considered to be *known*. The next result provides a sufficient condition on the number of samples, which guarantees that the $\mathcal{L}^{(2)}$ distance between the true distribution and the estimated distribution is small, and the last happens with probability at least $1 - \delta$.

Theorem 3.3 (Sufficient number of samples for inference) *Fix a number $\delta \in (0, 1)$ and $\eta > 0$ and let*

$$c_1(\eta, \beta, q) \triangleq \frac{512}{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \eta (1-2q)^4 \right]^2}, \quad (23)$$

$$c_2(\beta, q) \triangleq \frac{1152e^{2\beta}}{(1-2q)^4} \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta} \right)^2. \quad (24)$$

If the number of samples n_{\dagger} satisfies the inequality

$$n_{\dagger} > \max \{c_1(\eta, \beta, q), c_2(\beta, q)\} \log \left(\frac{4p^2}{\delta} \right), \quad (25)$$

then for the Chow-Liu-CC algorithm it is true that

$$\mathbb{P} \left(\mathcal{L}^{(2)} \left(p(\cdot), \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \right) \leq \eta \right) \geq 1 - \delta. \quad (26)$$

Conversely, the following result provides the necessary number of samples for small $\mathcal{L}^{(2)}$ distance by a minimax bound, which characterizes any possible estimator ψ . In other words, it provides the necessary number of samples required for accurate distribution estimation, appropriate for Predictive Learning (small $\mathcal{L}^{(2)}(\cdot)$).

Theorem 3.4 (Necessary number of samples for inference) *Fix a number $\delta \in (0, 1)$. Choose $\eta > 0$ such that $\tanh(\alpha) + 2\eta < \tanh(\beta)$. If the given number of samples satisfies the inequality*

$$n_{\dagger} < \frac{1 - [\tanh(\alpha) + 2\eta]^2}{16\eta^2[1 - (4q(1-q))^p]} \log p, \quad (27)$$

then for any algorithm ψ , it is true that

$$\inf_{\psi} \sup_{\substack{\mathbb{T} \in \mathcal{T} \\ \mathbb{P}(\cdot) \in \mathcal{P}_{\mathbb{T}}(\alpha, \beta)}} \mathbb{P}(\mathcal{L}^2(\mathbb{P}(\cdot), \psi(\mathbf{Y}_{1:n})) > \eta) > \frac{1}{2}.$$

Theorems 3.3 and 3.4 reduce to the noiseless setting for $q = 0$, which has been studied earlier by Bresler and Karzand (2018)². Similarly to our structure learning results, presented previously (Theorem 3.1, Theorem 3.2), when $q \rightarrow 1/2$ we have $n_{\dagger} \rightarrow \infty$, which indicates that the structure learning task becomes impossible for $q = 1/2$.

Remark 3.5 *Theorem 3.4 requires the assumption $\alpha < \beta$. The special case $\alpha = \beta$ can be derived by applying the same proof technique of Theorem 3.4 combined with Theorem B.1 by Bresler and Karzand (2018) and the SDPI by Polyanskiy and Wu (2017).*

Further details and proof sketches of Theorems 3.3 and 3.4 are provided in Section 4.3.

3.3 Estimating Higher Order Moments of Signed-Valued Trees

A collection of moments is sufficient to represent completely any probability mass function. For many distributions, the first and second order moments are sufficient statistics; this is true, for instance, for the *Gaussian distribution* or the *Ising model* with unitary and pairwise interactions. Even further, in the Gaussian case, the well-known Isserlis Theorem (Isserlis (1918)) gives a closed form expression for all moments of every order. As part of this work, we derive the corresponding moment expressions, *for any tree-structured Ising model*. To derive the expression of higher order moments, we first prove a key property of tree structures: for any tree structure $\mathbb{T} = (\mathcal{V}, \mathcal{E})$ and a even-sized set of nodes $\mathcal{V}' \subset \mathcal{V}$, we can partition \mathcal{V}' into $|\mathcal{V}'|/2$ pairs of nodes, such that the path along any pair is disjoint with the path of any other pair (see Appendix A, Lemma A.4). We denote as $\mathcal{C}_{\mathbb{T}}(\mathcal{V}')$ the set of distinct $|\mathcal{V}'|/2$ pairs of nodes in \mathcal{V}' , such that $\text{path}(u, u') \cap \text{path}(w, w') = \emptyset$, for all $\{u, u'\}, \{w, w'\} \in \mathcal{C}_{\mathbb{T}}(\mathcal{V}')$. Let $\mathcal{CP}_{\mathbb{T}}(\mathcal{V}')$ be the set of all edges in all mutually disjoint paths with endpoints the pairs of nodes in \mathcal{V}' , that is,

$$\mathcal{CP}_{\mathbb{T}}(\mathcal{V}') \triangleq \bigcup_{\{w, w'\} \in \mathcal{C}_{\mathbb{T}}(\mathcal{V}')} \text{path}_{\mathbb{T}}(w, w'). \quad (28)$$

For any tree \mathbb{T} , the set $\mathcal{CP}_{\mathbb{T}}(\mathcal{V}')$ can be computed via the Matching Pairs algorithm, Algorithm 2. By using the notation above, we can now present the equivalent of Isserlis' Theorem. The closed form expression of moments is given by the next theorem.

Theorem 3.6 *For any distribution of the form of (4), which factorizes according to a tree \mathbb{T} and has support $\{-1, +1\}^p$, it is true that*

$$\mathbb{E}[X_{i_1} X_{i_2} \dots X_{i_k}] = \begin{cases} 0 & k \text{ odd} \\ \prod_{e \in \mathcal{CP}_{\mathbb{T}}(i_1, i_2, \dots, i_k)} \mu_e & k \text{ even.} \end{cases} \quad (29)$$

2. We discuss further properties of the results and the source of the term $e \tanh(\beta) \log\left(\frac{1}{\tanh(\beta)}\right)$ of (23) in Section 4.4.

Algorithm 2 Matching Pairs

Require: Tree structure $T = (\mathcal{V}, \mathcal{E})$, any set $\mathcal{V}' \subset \mathcal{V} : |\mathcal{V}'| \in 2\mathbb{N}$

```

1:  $\mathcal{CP}_T \leftarrow \emptyset$ 
2: for  $i \in \mathcal{V}$  do
3:   if  $i \in \mathcal{V}'$  then
4:      $p(i) \leftarrow 1$ 
5:   else
6:      $p(i) \leftarrow 0$ 
7: Initialize each  $L(k)$  which containing all vertices in  $\mathcal{V}$  with depth  $k$  in the tree.
8: for  $k \in [d]$  do
9:   for  $i \in L(d - 1 + k)$  do
10:    if  $p(i) = 1$  then
11:       $\mathcal{CP}_T \leftarrow \mathcal{CP}_T \cup (i, \text{ancestor}(i))$ 
12:      if  $p(\text{ancestor}(i)) = 1$  then
13:         $\mathcal{V}' \leftarrow \mathcal{V}' \setminus \{i\}$ 
14:         $\mathcal{V}' \leftarrow \mathcal{V}' \setminus \{\text{ancestor}(i)\}$ 
15:         $p(\text{ancestor}(i)) \leftarrow 0$ 
16:      else
17:         $p(\text{ancestor}(i)) \leftarrow 1$ 
18:    if  $\mathcal{V}' \equiv \emptyset$  then
19:      return  $\mathcal{CP}_T$ 
    
```

Theorem 3.6 is an equivalent of Isserlis' theorem for tree-structured sign-valued distributions. Equation (29) is used later to define an estimator of higher order moments which requires two sufficient statistics: the estimated structure T_{\dagger}^{CL} and the correlations estimates $\hat{\mu}_e^{\dagger}$, for any $e \in T_{\dagger}^{\text{CL}}$. The higher order moments *altogether*, along with the parameter q , completely characterize the distribution of the noisy variables of the hidden model (13). The proof of Theorem 3.6 is provided in Appendix A.

High Order Moments Estimator: Consider any higher order moment as the expected value of the product of the hidden tree-structured Ising model variables $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ where $\mathcal{V}' \triangleq \{i_1, i_2, \dots, i_k\} \subset \mathcal{V}$. Theorem 3.6 gives the closed form solution of those moments and indicates the following estimator for higher order moments, considering that only noisy observations might be available, and q is known. In particular, we have

$$\hat{\mathbb{E}}[X_{i_1} X_{i_2} \dots X_{i_k}] \equiv 0, \quad k \in 2\mathbb{N} + 1, \quad (30)$$

$$\hat{\mathbb{E}}[X_{i_1} X_{i_2} \dots X_{i_k}] \triangleq \prod_{e \in \mathcal{CP}_{T_{\dagger}^{\text{CL}}}(i_1, i_2, \dots, i_k)} \frac{\hat{\mu}_e^{\dagger}}{(1 - 2q)^2}, \quad k \in 2\mathbb{N}. \quad (31)$$

Since the estimated structure and the pairwise correlations are sufficient statistics, then given those, (31) suggests a computationally efficient estimator for higher order moments. Algorithm 2 with input the estimated structure T_{\dagger}^{CL} returns the set $\mathcal{CP}_{T_{\dagger}^{\text{CL}}}$. Thus, by estimating T_{\dagger}^{CL} , $\mathcal{CP}_{T_{\dagger}^{\text{CL}}}$ and $\hat{\mu}_e^{\dagger}$ for any $e \in T_{\dagger}^{\text{CL}}$, we can in turn estimate any higher order

moment through (31). Considering the absolute estimation error, we have

$$\left| \hat{\mathbb{E}} \left[\prod_{s \in \mathcal{V}'} X_s \right] - \mathbb{E} \left[\prod_{s \in \mathcal{V}'} X_s \right] \right| \leq 2|\mathcal{V}'| \mathcal{L}^{(2)} \left(\mathbb{p}(\cdot), \Pi_{\mathbb{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \right). \quad (32)$$

Theorem 3.3 guarantees small "ssTV" and in combination with (32) gives an upper bound on the higher order moment estimate (31). In Section 4.5, we provide further details and discussion about Theorem 3.6, Algorithm 2, which computes the sets $\mathcal{CP}_{\mathbb{T}}(\mathcal{V}')$, $\mathcal{CP}_{\mathbb{T}_{\dagger}^{\text{CL}}}(\mathcal{V}')$, and the bound on the error of estimation (32).

So far we have studied the consistency of the estimator with respect to the $\mathcal{L}^{(2)}$. To generalize our results, we are now interested in finding sample complexity bounds for sufficiently small ϕ -divergences. This part seems to be challenging, however an interesting case is the KL-divergence, which is found in a variety of testing algorithms (see Daskalakis et al. (2018)). The next result gives a bound for the sufficient number of samples, which guarantees small symmetric KL divergence with high probability. Here, the notation $\mathbf{S}_{\text{KL}}(P||Q)$ is used for the symmetric KL divergence of two distributions P, Q and $\mathbf{S}_{\text{KL}}(P||Q) \triangleq \mathbf{D}_{\text{KL}}(P||Q) + \mathbf{D}_{\text{KL}}(Q||P)$. For any Ising model distributions P, Q of (3), with corresponding interaction parameters θ, θ' we have

$$\mathbf{S}_{\text{KL}}(\theta||\theta') \triangleq \mathbf{S}_{\text{KL}}(P||Q) = \sum_{s,t \in \mathcal{E}} (\theta_{st} - \theta'_{st}) (\mu_{st} - \mu'_{st}). \quad (33)$$

Theorem 3.7 (Upper Bounds for the Symmetric KL Divergence) *If the number of samples n_{\dagger} of \mathbf{Y} satisfies*

$$n_{\dagger} \geq 4 \frac{\beta^2(p-1)^2}{(1-2q)^4 \eta_s^2} \log \left(\frac{p^2}{\delta} \right), \quad (34)$$

then for $\mathbb{p}(\cdot) \in \mathcal{P}_{\mathbb{T}}(\alpha, \beta)$ we have

$$\mathbb{P} \left(\mathbf{S}_{\text{KL}} \left(\mathbb{p}(\cdot) || \Pi_{\mathbb{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \right) \leq \eta_s \right) \geq 1 - \delta, \quad (35)$$

where $\mathbb{T}_{\dagger}^{\text{CL}}$ is the Chow-Liu tree defined in (36) and the estimate $\Pi_{\mathbb{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger})$ is given by (21).

The asymptotic behavior of the bound in (34) was recently studied by Daskalakis et al. (2018). In that work, a set of testing algorithms are proposed and analyzed under the assumption of an Ising model with respect to trees and arbitrary graphs. Theorem 3.7 gives rise to possible extensions of testing algorithms to the hidden model setting, which is an interesting subject for future work.

4. Discussion

In this section, we present sketches of proofs, we compare our results with prior work, we further elaborate on Algorithm 1, Algorithm 2 and the error of higher order moment estimates. First, we discuss the convergence of the estimate $\mathbb{T}_{\dagger}^{\text{CL}}$ (Section 4.1). In section 4.2, we explain the connection between the hidden and noiseless settings on the tree structure

learning problem. Later, in Section 4.3, we present the analysis and a sketch of proof for Theorem 3.3, and in Section 4.4 we compare our results concerning *predictive learning* from noisy observations with the corresponding results of the noiseless setting (studied by Bresler and Karzand (2018)). Finally, in Section 4.5, we provide further details about Theorem 3.6, discussion about the Matching Pairs algorithm (Algorithm 2) and the accuracy of the proposed higher order moments estimator (31).

4.1 Estimating the Tree Structure \mathbb{T}

In this work, the structure learning algorithm is based on the classical Chow-Liu algorithm, and is summarized in Algorithm 1. We can express its output as

$$\mathbb{T}_{\dagger}^{\text{CL}} = \operatorname{argmax}_{\mathbb{T} \in \mathcal{T}} \sum_{(i,j) \in \mathcal{E}_{\mathbb{T}}} \left| \hat{\mu}_{i,j}^{\dagger} \right|, \quad (36)$$

where the last comes from a direct application of Lemma 11.2 by Bresler and Karzand (2018). The difference between Algorithm 1 and the Chow-Liu algorithm of the noiseless scheme is the use of noisy observations as input, since we consider a hidden model, whereas Bresler and Karzand (2018) assume that observations directly from the tree-structured model are available. Further, (36) shows the consistency of the estimate $\mathbb{T}_{\dagger}^{\text{CL}}$ for sufficiently large n_{\dagger} . The tree structure estimator $\mathbb{T}_{\dagger}^{\text{CL}}$ converges to \mathbb{T} when $n \rightarrow \infty$, since

$$\lim_{n \rightarrow \infty} \hat{\mu}_{i,j}^{\dagger} \stackrel{\text{a.s.}}{=} c^2 \mu_{i,j}. \quad (37)$$

From (36) and (37) we have (under an appropriate metric)

$$\lim_{n \rightarrow \infty} \mathbb{T}_{\dagger}^{\text{CL}} \stackrel{\text{a.s.}}{=} \mathbb{T}. \quad (38)$$

Asymptotically, both \mathbb{T}^{CL} and $\mathbb{T}_{\dagger}^{\text{CL}}$ converge to \mathbb{T} , where \mathbb{T}^{CL} denotes the structure estimate from noiseless data ($q = 0$). For a fixed probability of exact structure recovery $1 - \delta$, more samples are required in the hidden model setting, compared to the noiseless one. Additionally, the gap of the sample complexity between the noisy and noiseless setting comes from Theorem 3.1 by comparing the bound for the values $q = 0$ and $q \neq 0$.

4.2 Hidden Structure Recovery and Comparison with Prior Results

Theorem 3.1 and Theorem 3.2 constitute extensions of the noiseless setting, considered by Bresler and Karzand (2018, Theorem 3.2, Theorem 3.1), Bresler and Karzand (2018, Theorem 3.1, Theorem 3.2) to our hidden model; the noiseless results correspond to $q = 0$. In particular, in the presence of noise, *the dependence on p remains strictly logarithmic, that is, $\mathcal{O}(\log(p/\delta))$* . To make the connection between sufficient conditions more explicit, by setting $q = 0$ in (16) of Theorem 3.1, we retrieve the corresponding structure learning result by Bresler and Karzand (2018, Theorem 3.2) *exactly*: Fix a number $\delta \in (0, 1)$. If the number of samples of \mathbf{X} satisfy the inequality

$$n \geq \frac{32}{\tanh^2 \alpha (1 - \tanh \beta)} \log \left(\frac{2p^2}{\delta} \right), \quad (39)$$

then the Chow-Liu algorithm returns $\mathbf{T}^{\text{CL}} = \mathbf{T}$ with probability at least $1 - \delta$. An equivalent condition of (39) is

$$\tanh \alpha \geq \frac{4\epsilon}{\sqrt{1 - \tanh \beta}} \triangleq \tau(\epsilon), \text{ and } \epsilon \triangleq \sqrt{2/n \log(2p^2/\delta)}, \quad (40)$$

which shows that the weight of weakest edge should satisfy the following inequality $\alpha > \text{arctanh}(4\epsilon/\sqrt{1 - \tanh \beta})$ (Bresler and Karzand (2018)). For the hidden model, the equivalent extended condition for the weakest edge is

$$\tanh \alpha \geq \frac{4\epsilon_{\dagger} \sqrt{1 - (1 - 2q)^4 \tanh \beta}}{(1 - 2q)^2 (1 - \tanh \beta)} \triangleq \tau^{\dagger}(\epsilon_{\dagger}), \text{ and } \epsilon_{\dagger} = \sqrt{\frac{2 \log(2p^2/\delta)}{n_{\dagger}}}, \quad (41)$$

(see Appendix C, Lemma C.1) Condition (40) is retrieved through (41) for $q = 0$. Note that, for $q = 1/2$, the mutual information of the hidden and observable variables is zero, thus structure recovery is impossible.

Theorem 3.2 provides the necessary number of samples bound for exact structure recovery given noisy observations. In fact, it generalizes Theorem 3.1 by Bresler and Karzand (2018) to the hidden setting. By fixing $q = 0$, Theorem 3.2 recovers the noiseless case bound (Bresler and Karzand (2018), Theorem 3.1): Fix $\delta \in (0, 1)$. If the number of samples of \mathbf{X} satisfies the inequality

$$n < \frac{1}{16} e^{2\beta} [\alpha \tanh(\alpha)]^{-1} \log(p), \quad (42)$$

then for any algorithmic mapping (estimator) ψ , it is true that

$$\inf_{\psi} \sup_{\substack{\mathbf{T} \in \mathcal{T} \\ P \in \mathcal{P}_{\mathbf{T}}(\alpha, \beta)}} \mathbb{P}(\psi(X_{1:n}) \neq \mathbf{T}) > \frac{1}{2}. \quad (43)$$

When there is no noise, $q = 0$, we retrieve the noiseless result, while for any $q \in (0, 1/2)$ the sample complexity increases since $[1 - (4q(1 - q))^p]^{-1} > 1$ in (17) and for $q \rightarrow 1/2$ the required number of samples $n_{\dagger} \rightarrow \infty$, which makes structure learning impossible. The ratio between the noiseless and noisy necessary conditions indicates the gap between the hidden model and the original (noiseless) setting, which reads

$$\frac{n_{\dagger}}{n} \leq [1 - (4q(1 - q))^p]^{-1} \leq \frac{1}{\eta_{\text{KL}}}, \quad (44)$$

(see Appendix E.). The right hand-side of (44) is the strong data processing inequality for the binary symmetric channel, which was recently developed by Polyanskiy and Wu (2017, Equation (39)). We continue by providing the main idea and the important steps of the proof of Theorem 3.3.

4.3 Sufficient Events for Predictive Learning

The intersection of three events is sufficient to guarantee that $\mathcal{L}^{(2)}$ is upper bound by $\eta > 0$; those events are

$$\mathbf{E}_{\dagger}^{\text{corr}}(\epsilon_{\dagger}) \triangleq \left\{ \sup_{i,j \in \mathcal{V}} |\mu_{i,j}^{\dagger} - \hat{\mu}_{i,j}^{\dagger}| \leq \epsilon_{\dagger} \right\}, \quad (45)$$

$$\mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger) \triangleq \left\{ \left\{ (i, j) \in \mathcal{E}_T : |\tanh \theta_{ij}| \geq \frac{\tau^\dagger}{(1-2q)^2} \right\} \in \mathcal{E}_{T_\dagger^{\text{CL}}} \right\}, \quad (46)$$

$$\mathbf{E}_\dagger^{\text{cascade}}(\gamma_\dagger) \triangleq \left\{ \sup_{d \in [|\mathcal{E}|]} \left| \prod_{j=1}^d \hat{\mu}_j^\dagger - \prod_{j=1}^d \mu_j^\dagger \right| \leq \gamma_\dagger \right\}. \quad (47)$$

Our goal is to find the conditions on the parameters ϵ_\dagger , γ_\dagger which guarantee that the approximation error in the estimated distribution is less than a small positive number η , with probability at least $1 - \delta$, where δ is a fixed positive number.

In the noiseless scheme by (Bresler and Karzand, 2016, 2018) the events $\mathbf{E}^{\text{corr}}(\epsilon)$, $\mathbf{E}^{\text{strong}}(\epsilon)$ and $\mathbf{E}^{\text{cascade}}(\gamma)$ were defined in a similar manner; those are recovered through (45),(46), (47) for $q = 0$. Further, it is true that

$$\mathbb{P} \left[\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \right] \geq 1 - 2p^2 \exp \left(-\frac{n_\dagger \epsilon_\dagger^2}{2} \right), \quad (48)$$

$$\mathbb{P} \left[\mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger) \right] \geq 1 - 2p^2 \exp \left(-\frac{n_\dagger \epsilon_\dagger^2}{2} \right), \quad \text{and} \quad (49)$$

$$\mathbb{P} \left[\left| \prod_{r=1}^d \hat{\mu}_r^\dagger - \prod_{r=1}^d \mu_r^\dagger \right| \leq \gamma_\dagger \middle| \mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \right] = 1, \quad \forall \gamma_\dagger \geq \epsilon_\dagger d (1-2q)^{2(d-1)} \tanh^{d-1}(\beta), \quad (50)$$

where d is the length of a path in the hidden tree structure; thus, $d \leq |\mathcal{E}| = p - 1$. The function $f(d) = d(1-2q)^{2(d-1)} \tanh^{d-1}(\beta)$ has global maximum at $d = -1/\log((1-2q)^2 \tanh(\beta))$ which makes in the worst case, the maximum value of the difference $\left| \prod_{r=1}^d \hat{\mu}_r^\dagger - \prod_{r=1}^d \mu_r^\dagger \right|$ to be invariant of the longest path's size. The invariance over the length of paths has been shown by Bresler and Karzand (2018, Section E) in a different way. Our alternative approach is included in Appendix D (see Lemma D.2). For the complete proof of (48),(49),(50) see the Appendix (Lemma B.1, Lemma C.1 and Lemma D.1 respectively).

Additionally, the two events $\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \cap \mathbf{E}_\dagger^{\text{cascade}}(\gamma_\dagger) \cap \mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger)$ and $\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \cap \mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger)$ are equivalent, because $\mathbf{E}_\dagger^{\text{cascade}}(\gamma_\dagger) \subset \mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \cap \mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger)$ under a mild condition on γ_\dagger , (see Appendix D, Lemma D.1). Thus, (48) and (49) determine the sample complexity. Also, (50) shows that any assumption on ϵ_\dagger directly implies a corresponding assumption on γ_\dagger . The following sketch provides the required assumptions on the parameter ϵ_\dagger which directly affect the number of samples, in a similar way as in the noiseless case.

Sketch of Proof of Theorem 3.3: Our goal is to upper bound the ssTV of the true and the estimated distribution $\Pi_{T_\dagger^{\text{CL}}}(\hat{P}^\dagger)$ by a small number η . Similarly to Bresler's and Karzand's method, we start by using the triangle inequality, which gives the bound

$$\mathcal{L}^{(2)} \left(p(\cdot), \Pi_{T_\dagger^{\text{CL}}}(\hat{P}^\dagger) \right) \leq \mathcal{L}^{(2)} \left(\Pi_{T_\dagger^{\text{CL}}}(p(\cdot)), \Pi_{T_\dagger^{\text{CL}}}(\hat{P}^\dagger) \right) + \mathcal{L}^{(2)} \left(p(\cdot), \Pi_{T_\dagger^{\text{CL}}}(p(\cdot)) \right). \quad (51)$$

It is true that $\mathcal{L}^{(2)} \left(\Pi_{T_\dagger^{\text{CL}}}(p(\cdot)), \Pi_{T_\dagger^{\text{CL}}}(\hat{P}^\dagger) \right) \leq \eta/2$ under the event $\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger) \cap \mathbf{E}_\dagger^{\text{strong}}(\epsilon_\dagger) \cap \mathbf{E}_\dagger^{\text{cascade}}(\gamma_\dagger)$ when

$$\epsilon_{\dagger} \leq \min \left\{ \frac{e^{-\beta}(1-2q)^2}{20 \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta}\right)}, e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \frac{\eta}{3} (1-2q)^4 \right\}. \quad (52)$$

Furthermore, we have $\mathcal{L}^{(2)} \left(p(\cdot), \Pi_{T_{\dagger}^{\text{CL}}} (P) \right) \leq \eta/2$ under the event $E_{\dagger}^{\text{corr}}(\epsilon_{\dagger}) \cap E_{\dagger}^{\text{strong}}(\epsilon_{\dagger})$ when

$$\epsilon_{\dagger} \leq \min \left\{ \frac{\eta}{16} (1-2q)^2, \frac{(1-2q)^2 e^{-\beta}}{24 \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta}\right)} \right\}. \quad (53)$$

Notice that $e \tanh(\beta) \log [1/\tanh(\beta)] \leq 1$, for all $\beta \in \mathbb{R}^+$ and, as a consequence, it is sufficient to assume that

$$\epsilon_{\dagger} \leq \min \left\{ e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \frac{\eta}{16} (1-2q)^4, \frac{(1-2q)^2 e^{-\beta}}{24 \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta}\right)} \right\}. \quad (54)$$

The necessary assumption (54) on the parameter ϵ_{\dagger} , together with the expressions (48), (49), (50), give the required sample complexity (Theorem 3.3). In the next section, we compare our predictive learning results for the hidden model with those of the noiseless setting.

4.4 Predictive Learning, Comparison with Prior Results

Theorems 3.3 and 3.4 constitute hidden-model extensions of the noiseless setting (see Theorem 1.1 and Theorem 3.4 in the work of Bresler and Karzand (2018)). Here, we highlight the main differences between our work and that of Bresler and Karzand (2018). We first discuss the difference between the Chow-Liu and Chow-Liu-CC algorithms. Then we compare the necessary and sufficient number of samples with the corresponding results in the noiseless setting.

The Chow-Liu-CC Algorithm: *The first difference* lies in the structure estimation algorithm; the estimated structure T_{\dagger}^{CL} which is used for the predictive learning task is the output of the *clipped* Chow-Liu algorithm (Chow-Liu-CC) (see Algorithm 1). More specifically, we use the statistic T_{\dagger}^{CL} , provided by Algorithm 1, to construct the plug-in estimator $\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger})$. Algorithm 1 estimates the tree structure according to the classical Chow-Liu algorithm Bresler and Karzand (2018), *but* requires an extra step for the correlation estimates (case of Chow-Liu-CC algorithm), assuming that β and q are *known*. This step always reduces the error of estimation and simplifies the analysis.

Necessary number of samples: *Second*, Theorem 3 gives the required number of samples required for the observable random variable \mathbf{Y} , such that the $\mathcal{L}^{(2)}$ distance between the hidden model's distribution and the estimated distribution is small (less than $\eta > 0$), with probability greater than $1 - \delta$.

We highlight two facts. First, notice that the bound does not depend on α , because it is not necessary to recover correctly the weak edges (for the definition of weak-strong edges,

see (40),(41)). Second, the bound does not depend on the longest path of the hidden tree structure. This has been proved for the noiseless case by Bresler and Karzand (2018, Section E: End-to-end error on paths in the original tree, Lemma 8.7).

Our analysis shows that these two properties also hold for the generalized results of the noisy model. The main idea of Theorem 3.3 remains the same as in the noiseless setting. However, additional steps are necessary, which make the analysis challenging. In fact, we show that γ_{\dagger} is upper bounded by a function of ϵ_{\dagger} (see Appendix D, Lemma D.1). Thus, by applying any assumption on ϵ_{\dagger} , we directly apply a corresponding assumption on γ_{\dagger} , because of (50). As an extension of Bresler’s and Karzand’s method, we consider the sub-paths \mathcal{F}_i with arbitrary number of strong edges for each of them named as d_i , (see Appendix D, Lemma D.2). This extra step allows us to maximize with respect to the length of the path variable, named as d , and derive the necessary assumptions on ϵ_{\dagger} . These two modifications of the original method by Bresler and Karzand (2018, 6.1 Loss due to parameter estimation, page 13) produce the term $e \tanh(\beta) \log(1/\tanh(\beta))$, which is a necessary artifact of our approach. This term participates in $c_1(\eta, \beta, q)$, (23). For $q = 0$, it introduces a discrepancy between our final result and the corresponding result of the noiseless setting by Bresler and Karzand (2018).

As a supplement to the main results presented in this paper, we also investigate examine the cause of that discrepancy by extending (Bresler and Karzand (2018), Lemma E.1) and derive the corresponding result which reduces exactly to the noiseless setting bound for $q = 0$, (see Appendix D, Lemma F.1). The final result of this method includes a function with known properties but unknown closed form, which makes the result intractable. The final bound appears to be rather complicated and requires an extended analysis for its proof. Specifically, the dependence between the random variables μ_e^{\dagger} for $e \in \mathcal{E}_{\mathsf{T}}$ makes the problem much harder and allow us to derive a form which depends on a some bounded unknown function with a few known properties. On the contrary, our alternative approach (leading to our Theorem 3.2) results in a tractable lower bound on the sample complexity, which is strictly computable, and also provides a compact and understandable quantification of the dependence of n_{\dagger} on the parameters of the problem.

Sufficient number of samples: *Third*, Theorem 3.4 gives an upper bound for the sample complexity, such that no algorithm can guarantee small $\mathcal{L}^{(2)}$ with probability less than $1/2$. Our approach combines the Strong Data Processing Inequality by Polyanskiy and Wu (2017) with another method by Bresler and Karzand (2018, Theorem 3.4), to derive a non-trivial extension for the hidden model. For $q = 0$, we exactly recover the expression of the corresponding result in the noiseless case. On the other hand, for $q \rightarrow 1/2$ the number of samples $n_{\dagger} \rightarrow \infty$ and learning becomes impossible, as intuitively expected.

4.5 Estimating Higher Order Moments

Herein, we presented further necessary details regarding Theorem 3.6 and the *Matching Pairs* algorithm, which returns the set $\mathcal{CP}_{\mathsf{T}}(\mathcal{V}')$ (29). A short proof for the bound on the error of estimation (32) is given as the last part of this section.

Proof sketch of Theorem 3.6: We prove that $\mathcal{C}_{\mathsf{T}}(\mathcal{V}')$ always exists (when k is even) by induction (see Appendix A, Lemma A.4). We define the set of edges $\mathcal{CP}_{\mathsf{T}}(\mathcal{V}')$ as the union of

the disjoint paths³ $\mathcal{CP}_T(\mathcal{V}') = \cup_{w,w' \in \mathcal{C}_T(\mathcal{V}')} \text{path}(w, w')$. Combining the set $\mathcal{CP}_T(\mathcal{V}')$ together with the independent products property (see Lemma A.2), we derive the final expression (see Appendix A, proof of Theorem 3.6). Given the tree structure T and the correlations μ_e for all $e \in \mathcal{E}$, we can calculate the higher order expectations. Notice that the collection of disjoint paths \mathcal{CP}_T depends on the tree structure and as a consequence an algorithm is required to discover those paths. Different matching algorithms can be considered to find the set \mathcal{CP}_T . We propose Algorithm 2 which is simple and has low complexity $\mathcal{O}(|\mathcal{E}|)$.

Matching Pairs Algorithm: Algorithm 2 requires as input the tree and the set of nodes \mathcal{V}' which appear in the expectation, $\mathcal{V}' \equiv \{i_1, \dots, i_k\}$, and returns the set of edges $\mathcal{CP}_T(\mathcal{V}')$. For each node in the tree, a flag variable is assigned to each node and indicates if the corresponding node is a candidate for the final set $\mathcal{C}_T(\mathcal{V}')$ at the current step of the algorithm. The candidate nodes have to be matched with other nodes of the tree, such that the pairs generate disjoint paths. Initially, the candidate nodes are the nodes of the set \mathcal{V}' . Starting from the nodes which appear in the deepest level of the tree, we "move" them to their ancestor. At each step, if two candidate nodes appear at the same point, we match them as pair, we store the pair in the set $\mathcal{CP}_T(\mathcal{V}')$ and we remove both of them from the set \mathcal{V}' . We continue until $\mathcal{V}' \equiv \emptyset$. The complexity of Algorithm 2 is $\mathcal{O}(|\mathcal{E}|)$. Finally, Theorem 3.6 can be extended to any forest F structure by considering the set $\mathcal{CP}_F(\mathcal{V}')$ instead of $\mathcal{CP}_T(\mathcal{V}')$, where $\mathcal{CP}_F(\mathcal{V}') \triangleq \cup_i \mathcal{CP}_{T_i}(\mathcal{V}')$ and T_i is the i^{th} connected tree of the forest.

Estimation error of higher order moments: Inequality (32) bounds the error of estimation by the small set Total Variation, which is guaranteed to be less than $\eta > 0$ by Theorem 3.3. Additionally, the bound on the error of the estimation in (32) can be found as follows

$$\begin{aligned} & \left| \hat{\mathbb{E}} \left[\prod_{s \in \mathcal{V}'} X_s \right] - \mathbb{E} \left[\prod_{s \in \mathcal{V}'} X_s \right] \right| \\ &= \left| \prod_{e \in \mathcal{CP}_{T^\dagger}^{\text{CL}}(i_1, i_2, \dots, i_k)} \frac{\hat{\mu}_e^\dagger}{(1-2q)^2} - \prod_{e \in \mathcal{CP}_T(i_1, i_2, \dots, i_k)} \mu_e \right| \end{aligned} \quad (55)$$

$$\begin{aligned} &= \left| \prod_{e \in \mathcal{CP}_{T^\dagger}^{\text{CL}}(i_1, i_2, \dots, i_k)} \frac{\hat{\mu}_e^\dagger}{(1-2q)^2} - \prod_{e \in \mathcal{CP}_T(i_1, i_2, \dots, i_k)} \frac{\mu_e^\dagger}{(1-2q)^2} \right| \\ &= \left| \prod_{e \in \bigcup_{\{w, w'\} \in \mathcal{C}_{T^\dagger}^{\text{CL}}(\mathcal{V}')} \text{path}_{T^\dagger}^{\text{CL}}(w, w')} \frac{\hat{\mu}_e^\dagger}{(1-2q)^2} - \prod_{e \in \bigcup_{\{w, w'\} \in \mathcal{C}_T(\mathcal{V}')} \text{path}_T(w, w')} \frac{\mu_e^\dagger}{(1-2q)^2} \right| \end{aligned} \quad (56)$$

3. By disjoint paths we refer to paths with no common edges.

$$\begin{aligned}
 &= \left| \prod_{\{w,w'\} \in \mathcal{C}_{\mathbb{T}_{\dagger}^{\text{CL}}}(\mathcal{V}')} \prod_{e \in \text{path}_{\mathbb{T}_{\dagger}^{\text{CL}}}(w,w')} \frac{\hat{\mu}_e^{\dagger}}{(1-2q)^2} - \prod_{\{w,w'\} \in \mathcal{C}_{\mathbb{T}}(\mathcal{V}')} \prod_{e \in \text{path}_{\mathbb{T}}(w,w')} \frac{\mu_e^{\dagger}}{(1-2q)^2} \right| \\
 &\leq 2|\mathcal{V}'| \mathcal{L}^{(2)} \left(P, \Pi_{\mathbb{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \right), \tag{57}
 \end{aligned}$$

where (55) holds due to (29) and (31), (56) comes from (28) and the last inequality (57) is being proved by Bresler and Karzand (2018, Lemma 1, page 37). Thus, if we can accurately estimate the distribution under the sense $\mathcal{L}^{(2)} \left(P, \Pi_{\mathbb{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}) \right) \leq \eta'$, for a sufficiently small positive number η' , then by using (31) and choosing $\eta' \leq \eta/(2|\mathcal{V}'|)$, Theorem 3.3 guarantees accurate estimates for higher order moments with probability at least $1 - \delta$.

5. Conclusion

We have considered and analyzed the problem of predictive learning on hidden tree-structures from noisy observations, using the well-known Chow-Liu algorithm. In particular, we derived sample complexity guarantees for exact structure learning and distribution estimation. Our bounds constitute extensions of prior work (see Bresler and Karzand (2018)) to the hidden model, by introducing the cross-over probability q of the $\text{BSC}(q)^p$. Additionally, by applying a graph property for tree structures and a probabilistic property for Ising models, we derived an equivalent of the well-known Isserlis' theorem for Gaussian distributions, which yields to a consistent high-order moments estimator for Ising models.

Our results show that the estimated structure statistic $\mathbb{T}_{\dagger}^{\text{CL}}$ is essential for successful statistical inference on the hidden (or observable) layer, while the sample complexity with respect to number of nodes and probability of error remains strictly logarithmic, as in the noiseless case. Our hidden setting constitutes a first step towards more technically challenging and potentially more realistic statistical models, such as, for instance, structure and distribution learning when the noise is generated by an erasure channel, or when the underlying hidden tree structured distribution has a larger, or even uncountable support.

APPENDICES

The chart in Fig. 1 shows the various dependencies of the Lemmata or intermediate results either considered or developed in this paper, and the resulting Theorems. The proofs can be found in the corresponding section of the Appendix.

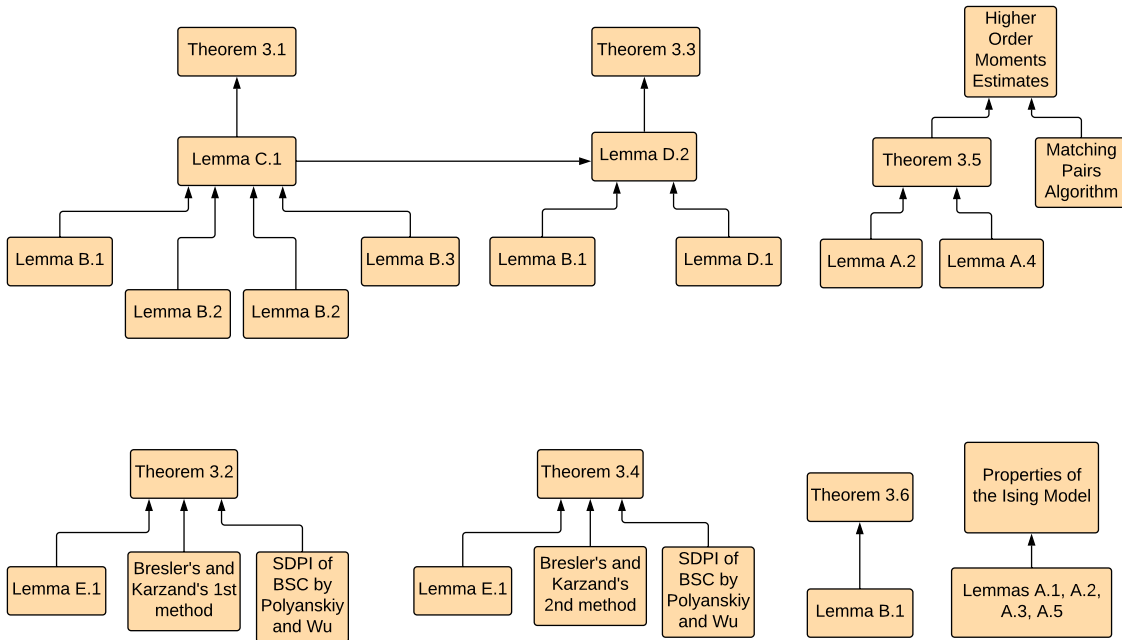


Figure 1: Stream mapping of the results

Appendix A. Properties of tree-shaped distributions

For completeness, we start with some properties which hold for any distribution with support $\{-1, +1\}^p$ and factorizes according to a tree structure (4) (Lauritzen, 1996). Later we derive explicit formulas for the Ising model (3).

Lemma A.1 *Any distribution $p(\mathbf{x})$ with respect to a forest $F = (\mathcal{V}, \mathcal{E})$, where $\mathbf{x} \in \{-1, 1\}^p$ and uniform marginals $\mathbb{P}(X_i = \pm 1) = 1/2$, for all $i \in \mathcal{V}$ can be expressed as*

$$p(\mathbf{x}) = \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2}. \tag{58}$$

Proof We prove the result for an arbitrary tree $T = (\mathcal{V}, \mathcal{E})$ and then we extend it to any forest structure by applying cuts to \mathcal{E} . The distribution factorizes according to the tree structure T and under the assumption of no external field (uniform marginal distributions), we have

$$\mathbb{P}(\mathbf{X} = \mathbf{x}) = \prod_{i \in \mathcal{V}} p(x_i) \prod_{(i,j) \in \mathcal{E}} \frac{p(x_i, x_j)}{p(x_i)p(x_j)}$$

$$\begin{aligned}
 &= \frac{1}{2^p} \prod_{(i,j) \in \mathcal{E}} \frac{\mathbb{P}(x_i, x_j)}{\left(\frac{1}{2}\right)^{p-1} \left(\frac{1}{2}\right)^{p-1}} \\
 &= 2^{p-2} \prod_{(i,j) \in \mathcal{E}} \mathbb{P}(x_i, x_j) \\
 &= 2^{p-2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{4} \tag{59}
 \end{aligned}$$

$$= \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2}, \tag{60}$$

where (59) holds since the joint distribution of any pair (X_i, X_j) of distinct nodes $i, j \in \mathcal{V}$ is

$$\mathbb{P}(x_i, x_j) = \mathbb{E}[\mathbf{1}_{X_i=x_i} \mathbf{1}_{X_j=x_j}] \stackrel{(2)}{=} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{4}. \tag{61}$$

The above result holds for the family of distributions with respect to any forest. By setting $\mathbb{E}[X_i X_j] = 0$ (cutting the (i, j)) for some $(i, j) \in \mathcal{E}$ we derive the distribution with respect to a forest generated by cutting the edge (i, j) of \mathbb{T} . \blacksquare

In Lemma A.2 we prove two fundamental properties of the model. These results were used by Bresler and Karzand (2018): for completeness we give a short and more general argument here. The first result is that if $\mathbb{P}(\mathbf{x})$ is a tree-shaped distribution on $\{-1, +1\}^p$ model with tree structure \mathbb{T} , then the random variables $\{X_i X_j : (i, j) \in \mathcal{E}\}$ are independent. The tree structure is necessary: the statement does not hold when there is at least one loop in the set of edges. The last fact can be proved by considering a simple counterexample with three nodes and a single cycle. The second part of Lemma A.2 is a short proof of the correlation decay property. To the best of our knowledge, these properties are known but there is not well known reference for the corresponding proofs in the literature.

Lemma A.2 *Let \mathbf{X} be a random binary vector in $\{-1, +1\}^p$ drawn according to a forest-structured distribution $\mathbb{P}(\cdot)$ with uniform marginal distributions on each entry X_i for $i \in [p]$. Then the elements of the collection of $|\mathcal{E}|$ random variables $\{X_i X_j : (i, j) \in \mathcal{E}\}$, are independent. Furthermore, we have*

$$\mathbb{E}[X_i X_j] = \prod_{e \in \text{path}(i,j)} \mu_e, \tag{62}$$

so the Correlation Decay Property (CDP) holds since $|\mu_e| \leq 1$ for all $e \in \mathcal{E}$.

Proof The following notation can be used to represent any acyclic graph. Let $(i_r)_{r=1}^p$ be an arbitrary permutation of $\ell = \{1, 2, \dots, p\}$. Notice that the singletons $\{i_r\}$, $r = 1, \dots, p$ form a partition of ℓ . Then, the set of edges \mathcal{E} is defined as

$$\mathcal{E} = (i_r, j_r)_{r=2}^p, \quad \text{and } j_1 = \emptyset, \quad j_r \in \{i_1, \dots, i_{r-1}\} \subset \ell. \tag{63}$$

As consequence we have that i_1 is the root of the tree since $j_1 = \emptyset$, from the definition of the j_r we can see that the graph with set of edges \mathcal{E} is an acyclic graph. Then it is sufficient to show that for any $\{c_r : r = 2, 3, \dots, p\} \in \{-1, +1\}^{p-1}$, it is true that

$$\mathbb{P}\left(\bigcap_{r=2}^p \{x_{i_r} x_{j_r} = c_r\}\right) = \prod_{r=2}^p \mathbb{P}(x_{i_r} x_{j_r} = c_r). \quad (64)$$

We have

$$\begin{aligned} \mathbb{P}\left(\bigcap_{r=2}^p \{X_{i_r} X_{j_r} = c_r\}\right) &= \sum_{\mathbf{x}: x_{i_r} x_{j_r} = c_r |_{r=2}^p} p(\mathbf{x}) \\ &= \sum_{\mathbf{x}: x_{i_r} x_{j_r} = c_r |_{r=2}^p} \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2} \end{aligned} \quad (65)$$

$$= \sum_{\mathbf{x}: x_{i_r} x_{j_r} = c_r |_{r=2}^p} \frac{1}{2} \prod_{r=2}^p \frac{1 + x_{i_r} x_{j_r} \mathbb{E}[X_{i_r} X_{j_r}]}{2} \quad (66)$$

$$\begin{aligned} &= \sum_{\mathbf{x}: x_{i_r} = c_r x_{j_r} |_{r=2}^p} \frac{1}{2} \prod_{r=2}^p \frac{1 + x_{i_r} x_{j_r} \mathbb{E}[X_{i_r} X_{j_r}]}{2} \\ &= \sum_{x_{i_1} \in \{-1, +1\}} \frac{1}{2} \prod_{r=2}^p \frac{1 + c_r \mathbb{E}[X_{i_r} X_{j_r}]}{2} \\ &= \prod_{r=2}^p \frac{1 + c_r \mathbb{E}[X_{i_r} X_{j_r}]}{2} \\ &= \prod_{r=2}^p \mathbb{P}(X_{i_r} X_{j_r} = c_r), \end{aligned} \quad (67)$$

(65) holds because of Lemma A.1 and (66) comes from (63). For the last equality we use the formula of the probability mass function of the variables $X_i X_j$. Notice that

$$\begin{aligned} p(x_i, x_j) &= \mathbb{E}\left[\left(\frac{1 + x_i X_i}{2}\right) \left(\frac{1 + x_j X_j}{2}\right)\right] \\ &= \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{4}, \end{aligned} \quad (68)$$

which holds for all $X_i, X_j \in \{-1, +1\}$ and gives the distribution of the product variable $X_{i_r} X_{j_r}$, which is

$$\mathbb{P}(X_{i_r} X_{j_r} = x_{i_r} x_{j_r}) = \frac{1 + x_{i_r} x_{j_r} \mathbb{E}[X_{i_r} X_{j_r}]}{2}. \quad (69)$$

Furthermore, for all $i, j \in \mathcal{V}$ there exists a unique path $\{i, k_1, k_2, \dots, k_\ell, j\}$ from i to j . Define the variable $\mathbb{1}_{(i,j)} \triangleq (X_{k_1} X_{k_1})(X_{k_2} X_{k_2}) \dots (X_{k_\ell} X_{k_\ell})$, which is equal to 1 almost surely, since $\mathbf{X} \in \{-1, +1\}$. $\mathbb{1}_{(\cdot)}$ should not be confused with $\mathbf{1}_A$, where the last denotes the indicator function of a set A . Then, we have

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_i \mathbb{1}_{(i,j)} X_j]$$

$$\begin{aligned}
 &= \mathbb{E}[X_i(X_{k_1}X_{k_1})(X_{k_2}X_{k_2}) \dots (X_{k_\ell}X_{k_\ell})X_j] \\
 &= \mathbb{E}[X_iX_{k_1}] \left(\prod_{m=1}^{\ell-1} \mathbb{E}[X_{k_m}X_{k_{m+1}}] \right) \mathbb{E}[X_\ell X_j] \tag{70}
 \end{aligned}$$

$$= \prod_{e \in \text{path}(i,j)} \mu_e, \tag{71}$$

and (70) comes from (64). For the first equality we used the fact that the random variable $\mathbb{1}_{(i,j)} \triangleq X_{k_1}X_{k_1}X_{k_2}X_{k_2} \dots X_{k_\ell}X_{k_\ell}$ is equal to 1 with probability 1. Notice that the correlation decay property (62) holds for any binary tree-structured model with support $\{-1, +1\}^p$ due to (64). \blacksquare

By studying a counter example for a graph with cycles, we can see that the products X_iX_j for $(i, j) \in \mathcal{E}$ are not independent for any two edges $(i, j), (i', j')$ which participate in the same cycle so the correlation decay property does not always hold for graphs with cycles. Examples for the correlations for graphs with cycles under a positive correlation assumption are given by Daskalakis et al. (2018).

The following lemma relates the pairwise correlations to the parameters of the Ising model.

Lemma A.3 *An equivalent expression of (3) is the following*

$$p(\mathbf{x}) = \frac{\prod_{(i,j) \in \mathcal{E}} [1 + x_i x_j \tanh(\theta_{ij})]}{\sum_{\mathbf{x}} \prod_{(i,j) \in \mathcal{E}} [1 + x_i x_j \tanh(\theta_{ij})]} \quad \mathbf{x} \in \{-1, 1\}^p. \tag{72}$$

Any tree-structured Ising model distribution (3) with zero-external field can be written as

$$p(\mathbf{x}) = \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \tanh(\theta_{ij})}{2}. \tag{73}$$

As a consequence from Lemma A.1 the correlations of an Ising model with respect to a tree $\Gamma = (\mathcal{V}, \mathcal{E})$ are

$$\mu_{i,j} = \mathbb{E}[X_i, X_j] = \tanh(\theta_{ij}), \quad \forall (i, j) \in \mathcal{E} \tag{74}$$

and

$$\mathbb{E}[X_i X_j] = \prod_{e \in \text{path}(i,j)} \mu_e = \prod_{e \in \text{path}(i,j)} \tanh(\theta_e), \tag{75}$$

so the correlation decay property (cdp) holds since $\mu_e \leq 1$ for all $e \in \mathcal{E}$.

Proof We can write $\exp(\theta_{ij}x_i x_j)$ as

$$\begin{aligned}
 \exp(\theta_{ij}x_i x_j) &= \frac{\exp(\theta_{ij}x_i x_j) + \exp(-\theta_{ij}x_i x_j)}{2} + \frac{\exp(\theta_{ij}x_i x_j) - \exp(-\theta_{ij}x_i x_j)}{2} \\
 &= \frac{\exp(\theta_{ij}) + \exp(-\theta_{ij})}{2} + x_i x_j \frac{\exp(\theta_{ij}) - \exp(-\theta_{ij})}{2} \tag{76}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\exp(\theta_{ij}) + \exp(-\theta_{ij})}{2} \left[1 + x_i x_j \frac{\exp(\theta_{ij}) - \exp(-\theta_{ij})}{\exp(\theta_{ij}) + \exp(-\theta_{ij})} \right] \\
 &= \cosh(\theta_{ij}) [1 + x_i x_j \tanh(\theta_{ij})], \tag{77}
 \end{aligned}$$

(76) holds because $x_i x_j \in \{-1, +1\}$. The partition function can be written as

$$\begin{aligned}
 Z(\theta) &= \sum_{\mathbf{x}} \prod_{(i,j) \in \mathcal{E}} \exp(\theta_{ij} x_i x_j) \\
 &= \sum_{\mathbf{x}} \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij}) [1 + x_i x_j \tanh(\theta_{ij})] \\
 &= \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij}) \sum_{\mathbf{x}} \prod_{(i,j) \in \mathcal{E}} [1 + x_i x_j \tanh(\theta_{ij})] \tag{78}
 \end{aligned}$$

$$= 2^p \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij}). \tag{79}$$

Notice that $\sum_{\mathbf{x}} \prod_{(i,j) \in \mathcal{E}} [1 + x_i x_j \tanh(\theta_{ij})] = 2^p$ under the assumption that the structure is a tree. The Ising model distribution with zero external field and with respect to a tree structure is

$$\begin{aligned}
 \mathbb{P}(\mathbf{X} = \mathbf{x}) &= \frac{\prod_{(i,j) \in \mathcal{E}} \exp(\theta_{ij} x_i x_j)}{Z(\theta)} \\
 &= \frac{\prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij}) [1 + x_i x_j \tanh(\theta_{ij})]}{2^p \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij})} \tag{80}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\prod_{(i,j) \in \mathcal{E}} [1 + x_i x_j \tanh(\theta_{ij})]}{2^p} \\
 &= \frac{1}{2} \prod_{(i,j) \in \mathcal{E}} \frac{1 + x_i x_j \tanh(\theta_{ij})}{2}, \tag{81}
 \end{aligned}$$

(77),(79) give (80) and $|\mathcal{E}| = p - 1$ gives (81). Notice that for any $p(\cdot)$ in (3), we have

$$\mathbb{E}[X_i X_j] = \frac{\partial \ln Z(\theta)}{\partial \theta_{ij}}, \quad \forall (i, j) \in \mathcal{E}, \tag{82}$$

under the tree assumption on the structure we have $Z(\theta) \stackrel{(79)}{=} 2^p \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij})$ and (82) gives

$$\begin{aligned}
 \mathbb{E}[X_i X_j] &= \frac{\partial \ln \left[2^p \prod_{(i,j) \in \mathcal{E}} \cosh(\theta_{ij}) \right]}{\partial \theta_{ij}} \\
 &= \tanh(\theta_{ij}), \quad \forall (i, j) \in \mathcal{E}. \tag{83}
 \end{aligned}$$

The proof is complete. ■

Lemma A.4 *Let \mathcal{V}' be a set of nodes such that $\mathcal{V}' \subset \mathcal{V}$ and $|\mathcal{V}'| \in 2\mathbb{N}$. Then it always exists a set $\mathcal{C}_T(\mathcal{V}')$ of $|\mathcal{V}'|/2$ pairs of nodes of \mathcal{V}' , such that any two distinct pairs (w, w') , (v, v') in $\mathcal{C}_T(\mathcal{V}')$ are pairwise disjoint (have no common edge), that is*

$$\text{path}_T(w, w') \cap \text{path}_T(v, v') = \emptyset, \quad \forall (w, w'), (v, v') \in \mathcal{C}_T(\mathcal{V}') : (w, w') \neq (v, v'). \quad (84)$$

Proof We prove the existence of $\mathcal{C}_T(\mathcal{V}')$ by contradiction: Assume that the two distinct paths $\text{path}_T(w, u')$, $\text{path}_T(u, w')$ share at least one edge. Let their common sub-path be $\text{path}_T(z, z')$, Figure 2. z and z' do not necessarily differ from w, w', u, u' . Notice that the common sub-path is unique, otherwise the graph would have a cycle. Then we can always consider the permutation of the endpoints which gives the disjoint paths $\text{path}_T(w, u)$ and $\text{path}_T(w', u')$. Now the paths $\text{path}_T(w, u)$ and $\text{path}_T(w', u')$ are disjoint, however it is possible that one of them or both, contain sub-paths with common edges. Then, we similarly proceed by removing the common sub-paths as previously. The set of common edges strictly decreases through the process, which terminates when there are only paths with no common edge. ■

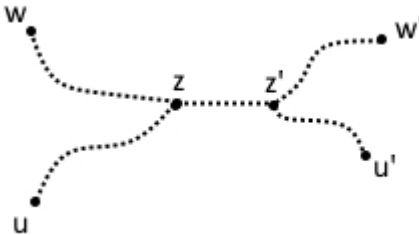


Figure 2: Proof of the existence of $\mathcal{C}_T(\mathcal{V}')$, Lemma A.4

Theorem 3.6 Assume $\mathbf{X} \sim p(\mathbf{x}) \in \mathcal{P}_T(\alpha, \beta)$, then

$$\mathbb{E}[X_{i_1} X_{i_2} \dots X_{i_k}] = \begin{cases} \prod_{e \in \mathcal{CP}_T(i_1, i_2, \dots, i_k)} \mu_e, & \forall k \in 2\mathbb{N} \\ 0, & \forall k \in 2\mathbb{N} + 1 \end{cases} \quad (85)$$

where the set of edges $\mathcal{CP}_T(i_1, \dots, i_k)$ is a collection of $k/2$ disjoint paths with endpoints pairs of the nodes i_1, \dots, i_k for each path. Given a tree structure T , $\mathcal{CP}_T(i_1, \dots, i_k)$ is found by running Algorithm 2 on T .

Proof We prove each case separately:

Even k . We proceed by showing that the Algorithm 2 returns the unique set of edges which gives the factorization of the higher order expectation as a product of the correlations μ_e . When $k=2$ the expression is proved in Lemma A.2. For $k > 2$ we have a product of correlations over edges in a collection of disjoint paths instead of one path. To prove the existence of the solution we have to prove the following statement, that is,

We continue by using Lemma A.4 in combination with Lemma A.2 to derive the expression of the higher order correlation when k is even. For all $i, j \in \mathcal{V}$ there exists a

unique path $\{i, k_1, k_2, \dots, k_\ell, j\}$ from i to j . Define as previously the variable $\mathbb{1}_{(i,j)} \triangleq (X_{k_1} X_{k_1})(X_{k_2} X_{k_2}) \dots (X_{k_\ell} X_{k_\ell})$, which is equal to 1 almost surely. Also, define the set of nodes $\mathcal{V}' \triangleq \{i_1, i_2, \dots, i_k\}$. If a collection of pairs of nodes $\mathcal{C}_T(\mathcal{V}')$ forms a set of disjoint paths, then the random variables $X_i \mathbb{1}_{(i,j)} X_j$ are independent for all the pairs $(i, j) \in \mathcal{C}_T(\mathcal{V}')$. Then without loss of generality we assume that the variables in the product $X_{i_1} X_{i_2} \dots X_{i_k}$ are ordered such such that the pairs $X_{i_j}, X_{i_{j+1}}$ for all $j \in \{1, 3, 5, \dots, k-1\} \triangleq [k-1]^{\text{odd}}$ form disjoint paths, that is,

$$\text{path}(i_j, i_{j+1}) \cap \text{path}(i_{j'}, i_{j'+1}) = \emptyset, \forall j \neq j' \in [k-1]^{\text{odd}}. \quad (86)$$

Then, we have

$$\mathbb{E}[X_{i_1} X_{i_2} \dots X_{i_k}] = \mathbb{E}[X_{i_1} \mathbb{1}_{(i_1, i_2)} X_{i_2} X_{i_3} \mathbb{1}_{(i_3, i_4)} X_{i_4} \dots X_{i_{k-1}} \mathbb{1}_{(i_{k-1}, i_k)} X_{i_k}] \quad (87)$$

$$= \prod_{j \in [k-1]^{\text{odd}}} \mathbb{E}[X_{i_j} \mathbb{1}_{(i_j, i_{j+1})} X_{i_{j+1}}] \quad (88)$$

$$= \prod_{j \in [k-1]^{\text{odd}}} \prod_{e \in \text{path}(i_j, i_{j+1})} \mu_e \quad (89)$$

$$= \prod_{e \in \mathcal{C}_T(i_1, i_2, \dots, i_k)} \mu_e, \quad (90)$$

where (87) and (88) come from (62), and (89) is true because of (86).

Odd k . From Lemma A.1 we have $p(\mathbf{x}) = \frac{1}{2} \prod_{(i,j) \in E} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2}$. Then

$$\begin{aligned} \mathbb{E}[X_1^{k_1} X_2^{k_2} \dots X_p^{k_p}] &= \frac{1}{2} \sum_{\mathbf{x} \in \{-1, +1\}^p} x_1^{k_1} x_2^{k_2} \dots x_p^{k_p} \prod_{(i,j) \in E} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2} \\ &= 0, \end{aligned} \quad (91)$$

for all $k_1, k_2, \dots, k_p \in \mathbb{N}$ such that $\sum_{i=1}^p k_i \in 2\mathbb{N} + 1$. \blacksquare

Lemma A.5 *For any binary random variables $X_i, X_j \in \{-1, +1\}^2$ with uniform marginals, the mutual information is*

$$I(X_i, X_j) = \frac{1}{2} \log_2 \left((1 - \mathbb{E}[X_i X_j])^{1 - \mathbb{E}[X_i X_j]} (1 + \mathbb{E}[X_i X_j])^{1 + \mathbb{E}[X_i X_j]} \right). \quad (92)$$

$I(X_i, X_j)$ is a symmetric function of $\mathbb{E}[X_i X_j]$ and increasing with respect to $|\mathbb{E}[X_i X_j]|$.

Proof The definition of mutual information gives

$$\begin{aligned} I(X_i, X_j) &= \sum_{x_i, x_j} p(x_i, x_j) \log_2 \frac{p(x_i, x_j)}{p(x_i) p(x_j)} \\ &= \sum_{x_i, x_j} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{4} \log_2 (1 + x_i x_j \mathbb{E}[X_i X_j]) \end{aligned} \quad (93)$$

$$\begin{aligned}
 &= \frac{1 + \mathbb{E}[X_i X_j]}{2} \log_2(1 + \mathbb{E}[X_i X_j]) + \frac{1 - \mathbb{E}[X_i X_j]}{2} \log_2(1 - \mathbb{E}[X_i X_j]) \\
 &= \frac{1}{2} \log_2 \left((1 - \mathbb{E}[X_i X_j])^{1 - \mathbb{E}[X_i X_j]} (1 + \mathbb{E}[X_i X_j])^{1 + \mathbb{E}[X_i X_j]} \right), \tag{94}
 \end{aligned}$$

and (61) gives (93). \blacksquare

Appendix B. Bounding the probability of mis-estimating correlations

The following lemma bounds the probability that the estimated pairwise correlations in the graph deviate from their true values. This follows from standard concentration of measure arguments.

Lemma B.1 *The event $E_{\dagger}^{corr}(\epsilon_{\dagger})$ defined in (45) holds with high probability,*

$$\mathbb{P}(E_{\dagger}^{corr}(\epsilon_{\dagger})) \geq 1 - p^2 \exp\left(\frac{-n_{\dagger} \epsilon_{\dagger}^2}{2}\right). \tag{95}$$

Proof Let $Z_{\dagger}^{(i)}$ be the i th sample of $Z_{\dagger} = Y_w Y_{\bar{w}} = N_w X_w N_{\bar{w}} X_{\bar{w}}$. Then $\hat{\mu}_{i,j}^{\dagger} = \frac{1}{n_{\dagger}} \sum_{i=1}^{n_{\dagger}} Z_{\dagger}^{(i)} = \frac{1}{n_{\dagger}} \sum_{i=1}^{n_{\dagger}} N_w^{(i)} X_w^{(i)} N_{\bar{w}}^{(i)} X_{\bar{w}}^{(i)}$ for all $i \neq j \in \mathcal{V}$. Then Hoeffding's inequality and union bound over all pairs of nodes $\binom{p}{2} < \frac{p^2}{2}$ give

$$\mathbb{P}(E_{\dagger}^{corr}(\epsilon_{\dagger})) \geq 1 - p^2 \exp\left(\frac{-n_{\dagger} \epsilon_{\dagger}^2}{2}\right), \tag{96}$$

completing the proof. \blacksquare

We apply Lemmata B.2, B.3, B.4 to Lemma C.1 to derive required number of samples for exact structure recovery when observations from a hidden model are given. To analyze the error event we use the "Two trees lemma" Bresler and Karzand (Bresler and Karzand, 2018, Lemma F.1). For two different spanning trees on same set of nodes. Informally, if two maximum spanning trees T, T' have a pair of nodes connected in a different way then there exist at least one edge in \mathcal{E}_T which does not exist in $\mathcal{E}_{T'}$ and vice versa. The formal expression of the statement and the corresponding proofs see Bresler and Karzand (2018) Lemma F.1: "Two Trees Lemma". Lemma B.2 analyzes the error characterization in Chow-Liu tree and it gives us the sufficient event which generates an error, $T_{\dagger}^{CL} \neq T$.

Lemma B.2 *Let $f = (w, \bar{w})$ be an edge such that, $f \in T$ and $f \notin T_{\dagger}^{CL}$ then there exists an edge $g \in T_{\dagger}^{CL}$ and $g \notin T$ such that $f \in \text{path}_T(u, \bar{u})$ and $g \in \text{path}_{T_{\dagger}^{CL}}(w, \bar{w})$ and the following holds under the error event $\{T \neq T_{\dagger}^{CL}\}$*

$$\left(\sum_{i=1}^{n_{\dagger}} Z_{f,u,\bar{u}}^{(i)} \right) \left(\sum_{i=1}^{n_{\dagger}} M_{f,u,\bar{u}}^{(i)} \right) < 0, \tag{97}$$

where $Z_{f,u,\bar{u}} \triangleq Y_w Y_{\bar{w}} - Y_u Y_{\bar{u}}$ and $M_{f,u,\bar{u}} \triangleq Y_w Y_{\bar{w}} + Y_u Y_{\bar{u}}$.

Proof Using similar approaches to the procedures as in Lemmata 8.2, 8.3 by Bresler and Karzand (2018) we have $|\hat{\mu}_f^\dagger| \leq |\hat{\mu}_g^\dagger| \implies$

$$\begin{aligned}
 0 &\geq |\hat{\mu}_f^\dagger|^2 - |\hat{\mu}_g^\dagger|^2 \\
 &= \left(\hat{\mu}_f^\dagger - \hat{\mu}_g^\dagger\right) \left(\hat{\mu}_f^\dagger + \hat{\mu}_g^\dagger\right) \\
 &= \frac{1}{n_\dagger^2} \left(\sum_{i=1}^{n_\dagger} N_w^{(i)} X_w^{(i)} N_{\bar{w}}^{(i)} X_{\bar{w}}^{(i)} - N_u^{(i)} X_u^{(i)} N_{\bar{u}}^{(i)} X_{\bar{u}}^{(i)} \right) \\
 &\quad \times \left(\sum_{i=1}^{n_\dagger} N_w^{(i)} X_w^{(i)} N_{\bar{w}}^{(i)} X_{\bar{w}}^{(i)} + N_w^{(i)} X_u^{(i)} N_{\bar{u}}^{(i)} X_{\bar{u}}^{(i)} \right) \\
 &= \frac{1}{n_\dagger^2} \left(\sum_{i=1}^{n_\dagger} Z_{f,u,\bar{u}}^{(i)} \right) \left(\sum_{i=1}^{n_\dagger} M_{f,u,\bar{u}}^{(i)} \right). \tag{98}
 \end{aligned}$$

The random variables $Z_{f,u,\bar{u}}^{(i)}$, $M_{f,u,\bar{u}}^{(i)}$ are functions of observations of the observable variables (noisy observations), while the method of the analysis by Bresler and Karzand (2018) remains the same. \blacksquare

In Lemmata (B.3), (B.4) we derive two concentration of measure inequalities for the variables $Z_{f,u,\bar{u}}^{(i)}$, $M_{f,u,\bar{u}}^{(i)}$. In fact we have that the event

$$\begin{aligned}
 E_Z \triangleq &\left\{ \left| \sum_{i=1}^{n_\dagger} Z_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[Z_{e,u,\bar{u}}] \right| \leq \right. \\
 &\left. n_\dagger \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 - \mu_A^\dagger} \right\} : \forall (w, \bar{w}) \in \mathcal{E} \text{ and } \forall u, \bar{u} \in \mathcal{V} \right\} \tag{99}
 \end{aligned}$$

happens with probability at least $1 - \frac{\delta'}{2}$ and the event

$$E_M \triangleq \left\{ \left| \sum_{i=1}^{n_\dagger} M_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[M_{e,u,\bar{u}}] \right| \leq \right. \tag{100}$$

$$\left. n_\dagger \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right\} : \forall (w, \bar{w}) \in \mathcal{E} \text{ and } \forall u, \bar{u} \in \mathcal{V} \right\} \tag{101}$$

happens with probability at least $1 - \frac{\delta''}{2}$. ϵ_\dagger is a decreasing function of n_\dagger , both ϵ_\dagger , μ_A are defined below. Finally, we apply union bound to guarantee that the event $E_Z \cup E_M$ happens with probability at least $1 - \delta$, where $\frac{\delta'}{2} + \frac{\delta''}{2} \leq 2 \max\{\frac{\delta'}{2}, \frac{\delta''}{2}\} \triangleq \delta$. The union bound is first applied over all pairs w, \bar{w}, u, \bar{u} in Lemmata B.3 and B.4 and then for the events E_Z, E_M .

Lemma B.3 *For all pairs of vertices $u, \bar{u} \in V$ and edges $e = (w, \bar{w})$ in the path $\text{path}_\Gamma(u, \bar{u})$ from u to \bar{u} , given n_\dagger samples $Z_{e,u,\bar{u}}^{(1)}, Z_{e,u,\bar{u}}^{(2)}, \dots, Z_{e,u,\bar{u}}^{(n)}$ of $Z_{e,u,\bar{u}} = Y_w Y_{\bar{w}} - Y_u Y_{\bar{u}}$, it is true that*

$$\mathbb{P} \left(\left| \sum_{i=1}^{n_\dagger} Z_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[Z_{e,u,\bar{u}}] \right| \leq n_\dagger \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 - \mu_A^\dagger} \right\} \right) \geq 1 - \frac{\delta}{2}, \tag{102}$$

where $\epsilon_{\dagger} = \sqrt{2/n_{\dagger} \log(2p^2/\delta)}$ and $A = \text{path}_{\mathbb{T}}(u, \bar{u}) \setminus \{e\}$.

Proof The proof is based on Bernstein's inequality (see Bennett (1962)). First, it is true that

$$\begin{aligned} Z_{e,u,\bar{u}} &= X_w N_w X_{\bar{w}} N_{\bar{w}} - N_u X_u N_{\bar{u}} X_{\bar{u}} \\ &= N_w X_w N_{\bar{w}} X_{\bar{w}} (1 - N_w X_w N_{\bar{w}} X_{\bar{w}} N_u X_u N_{\bar{u}} X_{\bar{u}}). \end{aligned} \quad (103)$$

Then,

$$\begin{aligned} \mathbb{E}[Z_{e,u,\bar{u}}] &= (1-2q)^2 \mathbb{E}[X_w X_{\bar{w}} - X_u X_{\bar{u}}] = (1-2q)^2 \mu_e (1 - \mu_A) \quad \text{and} \quad (104) \\ \text{Var}(Z_{e,u,\bar{u}}) &= \mathbb{E}\left[(Z_{e,u,\bar{u}})^2\right] - \mathbb{E}[Z_{e,u,\bar{u}}]^2 \\ &= \mathbb{E}\left[(X_w N_w X_{\bar{w}} N_{\bar{w}} - N_u X_u N_{\bar{u}} X_{\bar{u}})^2\right] - \left[(1-2q)^2 \mathbb{E}[X_w X_{\bar{w}} - X_u X_{\bar{u}}]\right]^2 \\ &= \mathbb{E}\left[1 + 1 - 2X_w N_w X_{\bar{w}} N_{\bar{w}} N_u X_u N_{\bar{u}} X_{\bar{u}}\right] - (1-2q)^4 \mathbb{E}[X_w X_{\bar{w}} - X_u X_{\bar{u}}]^2 \\ &= 2 - 2\mathbb{E}[X_w N_w X_{\bar{w}} N_{\bar{w}} N_u X_u N_{\bar{u}} X_{\bar{u}}] - (1-2q)^4 \mathbb{E}[X_w X_{\bar{w}} - X_u X_{\bar{u}}]^2 \\ &= 2 - 2(1-2q)^4 \mathbb{E}[X_w X_{\bar{w}} X_u X_{\bar{u}}] - (1-2q)^4 \mathbb{E}[X_w X_{\bar{w}} - X_u X_{\bar{u}}]^2 \\ &= 2 - 2(1-2q)^4 \mu_A - (1-2q)^4 (\mu_e (1 - \mu_A))^2 \\ &= 2 - (1-2q)^4 \left[2\mu_A + \mu_e^2 (1 - \mu_A)^2\right]. \end{aligned} \quad (105)$$

Using the expressions for the mean and the variance, we apply Bernstein's inequality (see Bennett (1962)) for the noisy setting: for all $i \in [n_{\dagger}]$ we have $\left|Z_{e,u,\bar{u}}^{(i)} - \mathbb{E}[Z_{e,u,\bar{u}}]\right| \leq M$ almost surely. Then, Bernstein's inequality gives

$$\begin{aligned} \mathbb{P}\left[\left|\sum_{i=1}^{n_{\dagger}} Z_{e,u,\bar{u}}^{(i)} - n_{\dagger} \mathbb{E}[Z_{e,u,\bar{u}}]\right| \geq t\right] &\leq 2 \exp\left(-\frac{t^2}{2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) + \frac{2}{3}Mt}\right), \quad \forall t > 0 \implies \\ \mathbb{P}\left[\left|\sum_{i=1}^{n_{\dagger}} Z_{e,u,\bar{u}}^{(i)} - n_{\dagger} \mathbb{E}[Z_{e,u,\bar{u}}]\right| \leq t\right] &\geq 1 - 2 \exp\left(-\frac{t^2}{2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) + \frac{2}{3}Mt}\right), \quad \forall t > 0. \end{aligned} \quad (106)$$

Choose a $\delta > 0$ and find t such that

$$\begin{aligned} \delta/2 &= 2 \exp\left(-\frac{t^2}{2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) + \frac{2}{3}Mt}\right) \implies \\ \log \frac{4}{\delta} &= \frac{t^2}{2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) + \frac{2}{3}Mt} \implies \\ t^2 - \frac{2}{3}Mt \log \frac{4}{\delta} - 2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) \log \frac{4}{\delta} &= 0 \implies \\ t_{1,2} &= \frac{\frac{2}{3}M \log \frac{4}{\delta} \pm \sqrt{\left(\frac{2}{3}M \log \frac{4}{\delta}\right)^2 + 8n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) \log \frac{4}{\delta}}}{2} \\ &= \frac{1}{3}M \log \frac{4}{\delta} \pm \sqrt{\left(\frac{1}{3}M \log \frac{4}{\delta}\right)^2 + 2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) \log \frac{4}{\delta}}. \end{aligned} \quad (107)$$

Since $t > 0$, we have

$$t = \frac{1}{3}M \log \frac{4}{\delta} + \sqrt{\left(\frac{1}{3}M \log \frac{4}{\delta}\right)^2 + 2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) \log \frac{4}{\delta}} \stackrel{M=4}{\implies} \quad (108)$$

$$t = \frac{4}{3} \log \frac{4}{\delta} + \sqrt{\left(\frac{4}{3} \log \frac{4}{\delta}\right)^2 + 2n_{\dagger} \text{Var}(Z_{e,u,\bar{u}}) \log \frac{4}{\delta}}. \quad (109)$$

If the probability of the union

$$\bigcup_{\forall u, \bar{u}, w, \bar{w}: (w, \bar{w}) \in \text{path}_{\mathbb{T}}(u, \bar{u})} \left\{ \left| \sum_{i=1}^{n_{\dagger}} Z_{e,u,\bar{u}}^{(i)} - n_{\dagger} \mathbb{E}[Z_{e,u,\bar{u}}] \right| \geq t \right\}$$

is at most $\frac{\delta}{2p^3}$, then the union bound gives probability at most $\frac{\delta}{2}$. Also,

$$\begin{aligned} \text{Var}(Z_{e,u,\bar{u}}) &= 2 - (1 - 2q)^4 \left[2\mu_A + \mu_e^2 (1 - \mu_A)^2 \right] \\ &= 2 - (1 - 2q)^4 2\mu_A - (1 - 2q)^4 \mu_e^2 (1 - \mu_A)^2 \\ &\leq 2 - (1 - 2q)^4 2\mu_A + 0 \\ &= 2 \left(1 - (1 - 2q)^4 \mu_A \right) \\ &= 2 \left(1 - \mu_A^{\dagger} \right). \end{aligned} \quad (110)$$

From (108) and (110), we have

$$\begin{aligned} t &= \frac{4}{3} \log \frac{4p^3}{\delta} + \sqrt{\left(\frac{4}{3} \log \frac{4p^3}{\delta}\right)^2 + 4n_{\dagger} \left(1 - \mu_A^{\dagger}\right) \log \frac{4p^3}{\delta}} \\ &\leq \frac{8}{3} \log \frac{4p^3}{\delta} + \sqrt{4n_{\dagger} \left(1 - \mu_A^{\dagger}\right) \log \frac{4p^3}{\delta}}, \end{aligned} \quad (111)$$

which implies that

$$\begin{aligned} t &= n_{\dagger} \left(\frac{4}{3n_{\dagger}} \log \frac{4p^3}{\delta} + \sqrt{\left(\frac{4}{3n_{\dagger}} \log \frac{4p^3}{\delta}\right)^2 + \frac{4}{n_{\dagger}} \left(1 - \mu_A^{\dagger}\right) \log \frac{4p^3}{\delta}} \right) \\ &\leq n_{\dagger} \left(\frac{8}{3n_{\dagger}} \log \frac{4p^3}{\delta} + \sqrt{\frac{4}{n_{\dagger}} \left(1 - \mu_A^{\dagger}\right) \log \frac{4p^3}{\delta}} \right). \end{aligned} \quad (112)$$

Define $\epsilon_{\dagger} = \sqrt{\log(2p^2/\delta) 2/n_{\dagger}}$ (as it is defined in Bresler and Karzand (2018)), then we get

$$t \leq n_{\dagger} \left(4\epsilon_{\dagger}^2 + 2\epsilon_{\dagger} \sqrt{1 - \mu_A^{\dagger}} \right) \leq n_{\dagger} \max \left\{ 8\epsilon_{\dagger}^2, 4\epsilon_{\dagger} \sqrt{1 - \mu_A^{\dagger}} \right\}. \quad (113)$$

Enough said. ■

Lemma B.4 gives the concentration of measure bound for the event E_M defined in (101).

Lemma B.4 For all pairs of vertices $u, \bar{u} \in V$ and edges $e = (w, \bar{w})$ in the path $\text{path}_T(u, \bar{u})$ from u to \bar{u} , given n_\dagger samples $M_{e,u,\bar{u}}^{(1)}, M_{e,u,\bar{u}}^{(2)}, \dots, M_{e,u,\bar{u}}^{(n)}$ of $M_{e,u,\bar{u}} = Y_w Y_{\bar{w}} + Y_u Y_{\bar{u}}$, it is true that

$$\mathbb{P} \left(\left| \sum_{i=1}^{n_\dagger} M_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[M_{e,u,\bar{u}}] \right| \leq n_\dagger \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right\} \right) \geq 1 - \frac{\delta}{2}, \quad (114)$$

where $\epsilon_\dagger = \sqrt{2/n_\dagger \log(2p^2/\delta)}$ and $A = \text{path}_T(u, \bar{u}) \setminus \{e\}$.

Proof Similarly to the prior Lemma, we calculate the mean and the variance as

$$\mathbb{E}[M_{e,u,\bar{u}}] = (1 - 2q)^2 \mathbb{E}[X_w X_{\bar{w}} + X_u X_{\bar{u}}] = (1 - 2q)^2 \mu_e (1 + \mu_A) \quad (115)$$

$$\begin{aligned} \text{Var}(M_{e,u,\bar{u}}) &= \mathbb{E} \left[(M_{e,u,\bar{u}})^2 \right] - \mathbb{E} \left[(M_{e,u,\bar{u}}) \right]^2 \\ &= \mathbb{E} \left[(X_w N_w X_{\bar{w}} N_{\bar{w}} + N_u X_u N_{\bar{u}} X_{\bar{u}})^2 \right] - \left[(1 - 2q)^2 \mathbb{E}[X_w X_{\bar{w}} + X_u X_{\bar{u}}] \right]^2 \\ &= \mathbb{E} \left[1 + 1 + 2X_w N_w X_{\bar{w}} N_{\bar{w}} N_u X_u N_{\bar{u}} X_{\bar{u}} \right] - (1 - 2q)^4 \mathbb{E}[X_w X_{\bar{w}} + X_u X_{\bar{u}}]^2 \\ &= 2 + 2\mathbb{E}[X_w N_w X_{\bar{w}} N_{\bar{w}} N_u X_u N_{\bar{u}} X_{\bar{u}}] - (1 - 2q)^4 \mathbb{E}[X_w X_{\bar{w}} + X_u X_{\bar{u}}]^2 \\ &= 2 + 2(1 - 2q)^4 \mathbb{E}[X_w X_{\bar{w}} X_u X_{\bar{u}}] - (1 - 2q)^4 \mathbb{E}[X_w X_{\bar{w}} + X_u X_{\bar{u}}]^2 \\ &= 2 + 2(1 - 2q)^4 \mu_A - (1 - 2q)^4 (\mu_e (1 + \mu_A))^2 \\ &= 2 + (1 - 2q)^4 \left[2\mu_A - \mu_e^2 (1 + \mu_A)^2 \right]. \end{aligned} \quad (116)$$

By applying Bernstein's inequality and we get that for any $t > 0$

$$\begin{aligned} \mathbb{P} \left[\left| \sum_{i=1}^{n_\dagger} M_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[M_{e,u,\bar{u}}] \right| \geq t \right] &\leq 2 \exp \left(-\frac{t^2}{2n_\dagger \text{Var}(M_{e,u,\bar{u}}) + \frac{2}{3}Mt} \right), \\ \mathbb{P} \left[\left| \sum_{i=1}^{n_\dagger} M_{e,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E}[M_{e,u,\bar{u}}] \right| \leq t \right] &\geq 1 - 2 \exp \left(-\frac{t^2}{2n_\dagger \text{Var}(M_{e,u,\bar{u}}) + \frac{2}{3}Mt} \right). \end{aligned} \quad (117)$$

Similarly, we find

$$t \leq n_\dagger \left(\frac{8}{3n_\dagger} \log \frac{4p^3}{\delta} + \sqrt{\frac{2}{n_\dagger} \text{Var}(M_{e,u,\bar{u}}) \log \frac{4p^3}{\delta}} \right) \quad (118)$$

and

$$\begin{aligned} \text{Var}(M_{e,u,\bar{u}}) &= 2 + (1 - 2q)^4 \left[2\mu_A - \mu_e^2 (1 + \mu_A)^2 \right] \\ &\leq 2 + (1 - 2q)^4 2\mu_A \\ &= 2 \left(1 + \mu_A^\dagger \right). \end{aligned} \quad (119)$$

We define $\epsilon_\dagger \triangleq \sqrt{\log(2p^2/\delta) 2/n_\dagger}$, then

$$t \leq n_\dagger \left(4\epsilon_\dagger^2 + 2\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right) \leq n_\dagger \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right\}, \quad (120)$$

which completes the proof. \blacksquare

Appendix C. Recovering strong edges

In Lemma C.1 we derive the set of strong edges for the hidden model. There is a threshold $\tau^\dagger \geq (1 - 2q)^2 \tau$, which determines the sufficiently strong edges, similarly to the noiseless setting by Bresler and Karzand (2018). Also we find a lower bound for the necessary number of samples for exact structure recovery. In fact we have $n_\dagger \geq n$, as expected. By setting $q = 0$ (then the probability to flip a bit equals to zero) we derive the exact expressions for the threshold τ and the sufficient number of samples defined in Bresler and Karzand (2018). Also in Lemma C.1 we show that event $E_\dagger^{\text{strong}}(\epsilon_\dagger)$ defined in (46) happens with probability at least $1 - \delta$. Under the event $E_\dagger^{\text{strong}}(\epsilon_\dagger)$ only the strong edges are guaranteed to exist in the estimated structure T_\dagger^{CL} .

Lemma C.1 *Define the set of strong edges: $\{(i, j) \in \mathcal{E}_T : |\tanh \theta_{ij}| \geq \frac{\tau^\dagger}{(1-2q)^2}\}$. Under the events defined in Lemmata B.3 and B.4 all the strong edges will be recovered from the Chow-Liu algorithm with probability at least $1 - \delta$. That is,*

$$\mathbb{P} \left[E_\dagger^{\text{strong}}(\epsilon_\dagger) \right] \geq 1 - \delta = 1 - 2p^2 \exp \left(-\frac{n_\dagger \epsilon_\dagger^2}{2} \right) \quad (121)$$

Proof Lemma B.2 gives

$$\begin{aligned} \left(\sum_{i=1}^{n_\dagger} Z_{f,u,\bar{u}}^{(i)} \right) \left(\sum_{i=1}^{n_\dagger} M_{f,u,\bar{u}}^{(i)} \right) < 0 &\implies \sum_{i=1}^{n_\dagger} Z_{f,u,\bar{u}}^{(i)} \leq 0 \text{ or } \sum_{i=1}^{n_\dagger} M_{f,u,\bar{u}}^{(i)} \leq 0 \implies \\ &\left| \sum_{i=1}^{n_\dagger} Z_{f,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E} \left[Z_{f,u,\bar{u}}^{(i)} \right] \right| \geq n_\dagger \mathbb{E} \left[Z_{f,u,\bar{u}}^{(i)} \right] \quad \text{or} \\ &\left| \sum_{i=1}^{n_\dagger} Y_{f,u,\bar{u}}^{(i)} - n_\dagger \mathbb{E} \left[Y_{f,u,\bar{u}}^{(i)} \right] \right| \geq n_\dagger \mathbb{E} \left[M_{f,u,\bar{u}}^{(i)} \right]. \end{aligned}$$

In addition, (104), (115), Lemma B.3 and Lemma B.4 give

$$\begin{aligned} (1 - 2q)^2 \mu_f (1 - \mu_A) &\leq \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 - \mu_A^\dagger} \right\} \quad \text{or} \\ (1 - 2q)^2 \mu_f (1 + \mu_A) &\leq \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right\} \quad \implies \end{aligned}$$

$$\begin{aligned} |\mu_f^\dagger| &\leq (1 - \mu_A)^{-1} \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 - \mu_A^\dagger} \right\} \quad \text{or} \\ |\mu_f^\dagger| &\leq (1 + \mu_A)^{-1} \max \left\{ 8\epsilon_\dagger^2, 4\epsilon_\dagger \sqrt{1 + \mu_A^\dagger} \right\} \quad \implies \end{aligned}$$

$$\begin{aligned}
 \left| \mu_f^\dagger \right| &\leq \max \left\{ \frac{8\epsilon_\dagger^2}{(1-\mu_A)}, \frac{8\epsilon_\dagger^2}{(1+\mu_A)}, \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)}, \frac{4\epsilon_\dagger\sqrt{1+\mu_A^\dagger}}{(1+\mu_A)} \right\} \implies \\
 \left| \mu_f^\dagger \right| &\leq \max \left\{ \frac{8\epsilon_\dagger^2}{(1-\mu_A)}, \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)} \right\} \implies \\
 \left| \mu_f^\dagger \right| &\leq \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)}. \tag{122}
 \end{aligned}$$

We get the last inequality for non trivial values of the bound $\frac{8\epsilon_\dagger^2}{(1-\mu_A^\dagger)} \leq 1$ and by using the following bound

$$\frac{8\epsilon_\dagger^2}{(1-\mu_A)} \leq \frac{16\epsilon_\dagger^2}{(1-\mu_A)} \leq \frac{4\epsilon_\dagger}{\sqrt{1-\mu_A}} = \frac{4\epsilon_\dagger\sqrt{1-\mu_A}}{(1-\mu_A)} \leq \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)}. \tag{123}$$

Finally the function $f(\mu_A) = \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)} = \frac{4\epsilon_\dagger\sqrt{1-(1-2q)^4\mu_A}}{(1-\mu_A)}$ is increasing with respect to μ_A (for all $\mu_A \leq 1$) and $\mu_A \leq \tanh \beta < 1$, we have

$$\left| \mu_f^\dagger \right| \leq \frac{4\epsilon_\dagger\sqrt{1-\mu_A^\dagger}}{(1-\mu_A)} \leq \frac{4\epsilon_\dagger\sqrt{1-(1-2q)^4 \tanh \beta}}{(1-\tanh \beta)} \triangleq \tau^\dagger \tag{124}$$

It is easy to verify that $\tau^\dagger > \tau = \frac{4\epsilon}{\sqrt{1-\tanh \beta}}$ when $n = n_\dagger$ (or $\epsilon = \epsilon_\dagger$). The weakest edge should satisfy the following property to guarantee the correct recovery of the tree under the event $E_\dagger^{\text{strong}}(\epsilon_\dagger)$

$$\begin{aligned}
 \left| \mu_f^\dagger \right| &\geq \tau^\dagger \implies \\
 (1-2q)^2 \tanh \alpha &\geq \frac{4\epsilon_\dagger\sqrt{1-(1-2q)^4 \tanh \beta}}{(1-\tanh \beta)} \implies \\
 \tanh \alpha &\geq \frac{4\epsilon_\dagger\sqrt{1-(1-2q)^4 \tanh \beta}}{(1-2q)^2(1-\tanh \beta)}, \quad q \in [0, \frac{1}{2}). \tag{125}
 \end{aligned}$$

The last inequality gives the definition of the strong edges in the noisy scheme. ■

In comparison to the noiseless setting (see Bresler and Karzand (2018)), we can guarantee exact recover with high probability under the event $E^{\text{strong}}(\epsilon)$ when the weakest edge satisfies the inequality

$$\tanh \alpha \geq \frac{4\epsilon}{\sqrt{1-\tanh \beta}}. \tag{126}$$

Notice that (126) can be obtained by (125) when $q = 0$ and $n = n_{\dagger}$. When $q > 0$ and $n = n_{\dagger}$ it is clear that the set of trees that can be recovered from noisy observations is a subset of the set of trees that can be recovered from the original observations. Also, we have

$$\begin{aligned}\epsilon &= \sqrt{2/n \log(2p^2/\delta)} \implies n = \frac{2}{\epsilon^2} \log(2p^2/\delta) \quad \text{and} \\ \epsilon_{\dagger} &= \sqrt{2/n_{\dagger} \log(2p^2/\delta)} \implies n_{\dagger} = \frac{2}{\epsilon_{\dagger}^2} \log(2p^2/\delta).\end{aligned}\tag{127}$$

By combining (125) with (127) we found the number of samples that we need to recover the tree with probability at $1 - \delta$ (Theorem 3.1):

$$n_{\dagger} > \frac{32 \left[1 - (1 - 2q)^4 \tanh \beta\right]}{(1 - \tanh \beta)^2 (1 - 2q)^4 \tanh^2 \alpha} \log \frac{2p^2}{\delta}.\tag{128}$$

On the other hand when there is no noise (Bresler and Karzand, 2018) we need

$$n > \frac{32}{\tanh^2 \alpha (1 - \tanh \beta)} \log \frac{2p^2}{\delta}.\tag{129}$$

Appendix D. Bounding the probability of cascades

Lemma D.1 *Under the event $E_{\dagger}^{corr}(\epsilon_{\dagger})$*

$$\mathbb{P} \left[\left| \prod_{r=1}^d \hat{\mu}_r^{\dagger} - \prod_{r=1}^d \mu_r^{\dagger} \right| \leq \gamma \middle| E_{\dagger}^{corr}(\epsilon_{\dagger}) \right] = 1, \quad \forall \gamma \geq \epsilon_{\dagger} d (1 - 2q)^{2(d-1)} \tanh^{d-1}(\beta),\tag{130}$$

for any $d \in [p - 1]$.

Proof We may write

$$\begin{aligned}\left| \prod_{r=1}^d \hat{\mu}_r^{\dagger} - \prod_{r=1}^d \mu_r^{\dagger} \right| &= \left| \sum_{r=1}^d \left(\hat{\mu}_r^{\dagger} - \mu_r^{\dagger} \right) \prod_{k=1}^{r-1} \hat{\mu}_k^{\dagger} \prod_{k=r+1}^d \mu_k^{\dagger} \right| \\ &\leq \sum_{r=1}^d \left| \left(\hat{\mu}_r^{\dagger} - \mu_r^{\dagger} \right) \prod_{k=1}^{r-1} \hat{\mu}_k^{\dagger} \prod_{k=r+1}^d \mu_k^{\dagger} \right| \\ &= \sum_{r=1}^d \left| \hat{\mu}_r^{\dagger} - \mu_r^{\dagger} \right| \prod_{k=1}^{r-1} \left| \hat{\mu}_k^{\dagger} \right| \prod_{k=r+1}^d \left| \mu_k^{\dagger} \right| \\ &\leq \sum_{r=1}^d \epsilon_{\dagger} \prod_{k=1}^{r-1} \left| \hat{\mu}_k^{\dagger} \right| \prod_{k=r+1}^d \left| \mu_k^{\dagger} \right|\end{aligned}\tag{131}$$

$$\leq \sum_{r=1}^d \epsilon_{\dagger} (1 - 2q)^{2(d-1)} \tanh^{d-1}(\beta)\tag{132}$$

$$\leq d(1-2q)^{2(d-1)} \tanh^{d-1}(\beta) \epsilon_{\dagger}, \quad (133)$$

(131) comes from Lemma B.1, (132) holds since $|\mu_k^\dagger| \leq (1-2q)^2 \tanh(\beta)$ and $|\hat{\mu}_k^\dagger| \leq (1-2q)^2 \tanh(\beta)$. In fact the probability of the estimates $|\hat{\mu}_k^\dagger|$ to be greater than the value $(1-2q)^2 \tanh(\beta)$ is not zero, however if we find any $|\hat{\mu}_k^\dagger|$ greater than $(1-2q)^2 \tanh(\beta)$, we set its value to $(1-2q)^2 \tanh(\beta)$. This policy decreases the estimation error, by using the fact that β and q are known. \blacksquare

D.1 Proof of Theorem 3.3, Predictive Learning in Hidden Sign-Valued Trees

The goal is to estimate the distribution $p(\cdot)$ when observations of Y are given. The estimated distribution is composed by the Chow-Liu tree T_{\dagger}^{CL} and a matching moment technique Chow and Liu (1968). In this section, we derive the corresponding statement of Propositions 5.1 and 5.2 in Bresler and Karzand (2018) for the hidden model. We find the estimate $\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)$ of P such that $\mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)) \leq \eta$ with high probability. Under the event $E_{\dagger}^{\text{corr}}(\epsilon_{\dagger}) \cap E_{\dagger}^{\text{strong}}(\epsilon_{\dagger}) \cap E_{\dagger}^{\text{cascade}}(\gamma_{\dagger})$ and for sufficiently small $\epsilon_{\dagger}, \gamma_{\dagger}$ we have $\mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)) \leq \eta$ with probability at least $1 - \delta$. The required upper bounds for the parameters $\epsilon_{\dagger}, \gamma_{\dagger}$ give the a lower bound for the number of samples n_{\dagger} . We derive a lower bound for n_{\dagger} in the following lemmata.

Lemma D.2 *With probability at least $1 - \delta$ we have $\mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)) \leq \eta$ under the assumption*

$$\epsilon_{\dagger} \leq \min \left\{ e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \frac{\eta}{16} (1-2q)^4, \frac{(1-2q)^2 e^{-\beta}}{24 (1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta})} \right\}.$$

Proof To upper bound the quantity $\mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger))$, we use the triangle inequality as

$$\mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)) \leq \mathcal{L}^{(2)}(P, \Pi_{T_{\dagger}^{\text{CL}}}(P)) + \mathcal{L}^{(2)}(\Pi_{T_{\dagger}^{\text{CL}}}(P), \Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger)). \quad (134)$$

The distribution $\Pi_{T_{\dagger}^{\text{CL}}}(P)$ can be written as

$$\begin{aligned} \Pi_{T_{\dagger}^{\text{CL}}}(P) &= \frac{1}{2} \prod_{(i,j) \in \mathcal{E}_{T_{\dagger}^{\text{CL}}}} \frac{1 + x_i x_j \mathbb{E}[X_i X_j]}{2} \\ &= \frac{1}{2} \prod_{(i,j) \in \mathcal{E}_{T_{\dagger}^{\text{CL}}}} \frac{1 + x_i x_j \frac{\mathbb{E}[Y_i Y_j]}{(1-2q)^2}}{2} \end{aligned} \quad (135)$$

where the last comes from (12), and

$$\Pi_{T_{\dagger}^{\text{CL}}}(\hat{P}^\dagger) \triangleq \frac{1}{2} \prod_{(i,j) \in \mathcal{E}_{T_{\dagger}^{\text{CL}}}} \frac{1 + x_i x_j \frac{\hat{\mathbb{E}}[Y_i Y_j]}{(1-2q)^2}}{2}. \quad (136)$$

For any tree-structured Ising model distributions P, \tilde{P} with structures $\mathsf{T} = (\mathcal{V}, \mathcal{E})$ and $\tilde{\mathsf{T}} = (\mathcal{V}, \tilde{\mathcal{E}})$ respectively, we have

$$\mathcal{L}^{(2)}(P, \tilde{P}) \triangleq \sup_{i,j \in \mathcal{V}} \frac{1}{2} \sum_{x_i, x_j \in \{-1, +1\}^2} \left| P(x_i, x_j) - \tilde{P}(x_i, x_j) \right| \quad (137)$$

$$= \sup_{i,j \in \mathcal{V}} \frac{1}{2} \left| \prod_{e \in \text{path}_{\mathsf{T}}(i,j)} \mu_e - \prod_{e' \in \text{path}_{\tilde{\mathsf{T}}}(i,j)} \tilde{\mu}_{e'} \right|, \quad (138)$$

(6) and (61) give (138), $\mu_e = \mathbb{E}_P[X_i X_j]$, where $e = (i, j) \in \mathcal{E}$ and $\tilde{\mu}_{e'} = \mathbb{E}_{\tilde{P}}[X_i X_j]$, where $e' = (i, j) \in \tilde{\mathcal{E}}$. By using the expression (138), we will find the necessary assumptions on ϵ_{\dagger} and γ_{\dagger} such that $\mathcal{L}^{(2)}(P, \Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(P)) \leq \eta/2$ and $\mathcal{L}^{(2)}(\Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(P), \Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger})) \leq \eta/2$. The first quantity is upper bounded by using Lemma 7.2, the second $\mathcal{L}^{(2)}(\Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(P), \Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger}))$ as

$$\mathcal{L}^{(2)}(\Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(P), \Pi_{\mathsf{T}_{\dagger}^{\text{CL}}}(\hat{P}^{\dagger})) = \sup_{i,j \in \mathcal{V}} \frac{1}{2} \left| \prod_{e \in \text{path}_{\mathsf{T}_{\dagger}^{\text{CL}}}(i,j)} \frac{\mu_e^{\dagger}}{(1-2q)^2} - \prod_{e \in \text{path}_{\mathsf{T}_{\dagger}^{\text{CL}}}(i,j)} \frac{\hat{\mu}_e^{\dagger}}{(1-2q)^2} \right|. \quad (139)$$

Our goal is to find the necessary assumptions on the parameters $\epsilon_{\dagger}, \gamma_{\dagger}$ such that the quantity in (139) is upper bounded by η . As an intermediate step we have to find an upper bound on τ^{\dagger} . For the noiseless setting we had

$$\tau \triangleq \frac{4\epsilon}{\sqrt{1 - \tanh \beta}} \leq 4\epsilon e^{\beta}, \text{ since } 1 - \tanh \beta \geq e^{-2\beta}. \quad (140)$$

Lemma C.1 gives

$$\begin{aligned} \tau^{\dagger} &= \frac{4\epsilon_{\dagger} \sqrt{1 - (1-2q)^4 \tanh \beta}}{(1 - \tanh \beta)} \\ &= \frac{4\epsilon_{\dagger} \sqrt{1 - (1-8q + 24q^2 - 32q^3 + 16q^4) \tanh \beta}}{(1 - \tanh \beta)} \\ &\leq 4\epsilon_{\dagger} \frac{\sqrt{1 - \tanh \beta} + \sqrt{(1-3q + 4q^2 - 2q^3) 8q \tanh \beta}}{(1 - \tanh \beta)} \\ &\leq 4\epsilon_{\dagger} e^{\beta} \left(1 + e^{\beta} \sqrt{(1-q)(2q^2 - 2q + 1) 8q \tanh \beta} \right) \end{aligned} \quad (141)$$

$$< 4\epsilon_{\dagger} e^{\beta} \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta} \right). \quad (142)$$

(140) gives (141). The last inequality (142) holds since $2q^2 - 2q + 1 \in [\frac{1}{2}, 1] \quad \forall q \in [0, \frac{1}{2}]$. Let t be the number of weak edges. In the case where the variable $\mathbf{X} \sim \mathfrak{p}(\cdot) \in \mathcal{P}_{\mathsf{T}}(\alpha, \beta)$ is observable, Bresler and Karzand found the necessary assumptions on ϵ and γ (Bresler and Karzand, 2018) such that

$$\mathcal{L}^{(2)}(\Pi_{\mathsf{T}^{\text{CL}}}(P), \Pi_{\mathsf{T}^{\text{CL}}}(\hat{P})) \leq \frac{\eta}{2}. \quad (143)$$

We follow a similar technique, to discover the necessary assumptions that we need for the parameters γ_{\dagger} and ϵ_{\dagger} . For our approach, Lemma D.1 is crucial, since it shows the worst path scenario through the the following analysis. Like Bresler's and Karzand's method, in the hidden model scheme, the path $\text{path}_{\text{TCL}}(i, j)$ considered is $(\mathcal{F}_0, e_1, \mathcal{F}_1, e_1, \dots, \mathcal{F}_{t-1}, e_t, \mathcal{F}_t)$, where \mathcal{F}_i are segments with all strong edges and e_i are all weak edges. It is necessary to denote the length of each segment \mathcal{F}_i explicitly, for all $i \in \{0, 1, \dots, t\}$. Each segment (sub-path) \mathcal{F}_i has exactly d_i edges, and the total number of edges in the path are d ; thus $d = \sum_{i=0}^t d_i + t$. We will use the inequalities $|\hat{\mu}_{\mathcal{F}_i}^{\dagger} - \mu_{\mathcal{F}_i}^{\dagger}| \leq \gamma_{\dagger}^{(i)}$, where $\gamma_{\dagger}^{(i)} = \epsilon_{\dagger} d_i (1 - 2q)^{2(d_i-1)} \tanh^{d_i-1}(\beta)$ under the event $E_{\dagger}^{\text{corr}}(\epsilon_{\dagger})$ (Lemma D.1) and $|\hat{\mu}_{e_i}^{\dagger} - \mu_{e_i}^{\dagger}| \leq \epsilon_{\dagger}$ under the event $E_{\dagger}^{\text{corr}}(\epsilon_{\dagger})$. Under these considerations, (139) may be upper-bounded as

$$\begin{aligned}
 & \left| \prod_{e \in \text{path}_{\text{TCL}}(i, j)} \frac{\mu_e^{\dagger}}{(1-2q)^2} - \prod_{e \in \text{path}_{\text{TCL}}(i, j)} \frac{\hat{\mu}_e^{\dagger}}{(1-2q)^2} \right| \\
 & \stackrel{(a)}{=} \frac{1}{(1-2q)^{2d}} \left| \hat{\mu}_{\mathcal{F}_0}^{\dagger} \prod_{i=1}^t \hat{\mu}_{\mathcal{F}_i}^{\dagger} \hat{\mu}_{e_i}^{\dagger} - \mu_{\mathcal{F}_0}^{\dagger} \prod_{i=1}^t \mu_{\mathcal{F}_i}^{\dagger} \mu_{e_i}^{\dagger} \right| \\
 & \stackrel{(b)}{\leq} \frac{1}{(1-2q)^{2d}} \left[\left| \hat{\mu}_{\mathcal{F}_0}^{\dagger} - \mu_{\mathcal{F}_0}^{\dagger} \right| \prod_{j=1}^t \left| \mu_{\mathcal{F}_j}^{\dagger} \mu_{e_j}^{\dagger} \right| \right. \\
 & \quad \left. + \sum_{i=1}^t \left| \hat{\mu}_{\mathcal{F}_i}^{\dagger} \hat{\mu}_{e_i}^{\dagger} - \mu_{\mathcal{F}_i}^{\dagger} \mu_{e_i}^{\dagger} \right| \left| \hat{\mu}_{\mathcal{F}_0}^{\dagger} \right| \prod_{j=1}^{i-1} \left| \hat{\mu}_{\mathcal{F}_j}^{\dagger} \hat{\mu}_{e_j}^{\dagger} \right| \prod_{k=i+1}^t \left| \mu_{\mathcal{F}_k}^{\dagger} \mu_{e_k}^{\dagger} \right| \right] \\
 & \stackrel{(c)}{\leq} \frac{1}{(1-2q)^{2d}} \left[\gamma_{\dagger}^{(0)} (1-2q)^{2(d-d_0-t)} (\tau^{\dagger})^t \right. \\
 & \quad \left. + (1-2q)^{2d} (\tau^{\dagger} + \epsilon_{\dagger})^{t-1} \sum_{i=1}^t (1-2q)^{-2(d_i+t)} \left| \hat{\mu}_{\mathcal{F}_i}^{\dagger} \hat{\mu}_{e_i}^{\dagger} - \mu_{\mathcal{F}_i}^{\dagger} \mu_{e_i}^{\dagger} \right| \right] \\
 & \stackrel{(d)}{\leq} \left[\frac{\gamma_{\dagger}^{(0)}}{(1-2q)^{2d_0}} \left(\frac{\tau^{\dagger}}{(1-2q)^2} \right)^t + \sum_{i=1}^t \frac{(\tau^{\dagger} + \epsilon_{\dagger})^{t-1}}{(1-2q)^{2(d_i+t)}} \left(\left| (\hat{\mu}_{\mathcal{F}_i}^{\dagger} - \mu_{\mathcal{F}_i}^{\dagger}) \hat{\mu}_{e_i}^{\dagger} \right| + \left| \mu_{\mathcal{F}_i}^{\dagger} (\hat{\mu}_{e_i}^{\dagger} - \mu_{e_i}^{\dagger}) \right| \right) \right] \\
 & \stackrel{(e)}{\leq} \left[\frac{\gamma_{\dagger}^{(0)}}{(1-2q)^{2d_0}} \left(\frac{\tau^{\dagger}}{(1-2q)^2} \right)^t + \sum_{i=1}^t \frac{(\tau^{\dagger} + \epsilon_{\dagger})^{t-1}}{(1-2q)^{2(d_i+t)}} \left(\gamma_{\dagger}^{(i)} + (1-2q)^{2d_i} \epsilon_{\dagger} \right) \right] \\
 & = \left[\frac{\gamma_{\dagger}^{(0)}}{(1-2q)^{2d_0}} \left(\frac{\tau^{\dagger}}{(1-2q)^2} \right)^t + \left(\frac{\tau^{\dagger} + \epsilon_{\dagger}}{(1-2q)^2} \right)^{t-1} \sum_{i=1}^t \left(\frac{\gamma_{\dagger}^{(i)}}{(1-2q)^{2(d_i+1)}} + \frac{\epsilon_{\dagger}}{(1-2q)^2} \right) \right] \\
 & \stackrel{(f)}{\leq} \left[\epsilon_{\dagger} d_0 \frac{\tanh^{d_0-1}(\beta)}{(1-2q)^2} \left(\frac{\tau^{\dagger}}{(1-2q)^2} \right)^t + \left(\frac{\tau^{\dagger} + \epsilon_{\dagger}}{(1-2q)^2} \right)^{t-1} \sum_{i=1}^t \left(\epsilon_{\dagger} d_i \frac{\tanh^{d_i-1}(\beta)}{(1-2q)^4} + \frac{\epsilon_{\dagger}}{(1-2q)^2} \right) \right] \\
 & \stackrel{(g)}{\leq} \left[\epsilon_{\dagger} d_0 \frac{\tanh^{d_0-1}(\beta)}{(1-2q)^2} \left(\frac{\tau^{\dagger} + \epsilon_{\dagger}}{(1-2q)^2} \right)^{t-1} + \left(\frac{\tau^{\dagger} + \epsilon_{\dagger}}{(1-2q)^2} \right)^{t-1} \sum_{i=1}^t \left(\epsilon_{\dagger} d_i \frac{\tanh^{d_i-1}(\beta)}{(1-2q)^4} + \frac{\epsilon_{\dagger}}{(1-2q)^2} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(h)}{\leq} \left(\frac{\tau^\dagger + \epsilon_\dagger}{(1-2q)^2} \right)^{t-1} \left[\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^2} \right. \\
 & \quad \left. + \sum_{i=1}^t \left(\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} + \frac{\epsilon_\dagger}{(1-2q)^2} \right) \right] \\
 & = \left(\frac{\tau^\dagger + \epsilon_\dagger}{(1-2q)^2} \right)^{t-1} \left[\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^2} \right. \\
 & \quad \left. + t \frac{\epsilon_\dagger}{(1-2q)^2} \left(\frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^2} + 1 \right) \right] \\
 & \stackrel{(i)}{\leq} \left(\frac{\tau^\dagger + \epsilon_\dagger}{(1-2q)^2} \right)^{t-1} \\
 & \quad \times \left[\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^2} + 2t \frac{\epsilon_\dagger}{(1-2q)^2} \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^2} \right] \\
 & \leq \left(\frac{\tau^\dagger + \epsilon_\dagger}{(1-2q)^2} \right)^{t-1} \left[\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} \right] (2t+1) \\
 & \stackrel{(j)}{\leq} \left(\frac{4\epsilon_\dagger e^\beta (1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta}) + \epsilon_\dagger}{(1-2q)^2} \right)^{t-1} (2t+1) \epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} \\
 & \leq \left(\frac{5\epsilon_\dagger e^\beta}{(1-2q)^2} \right)^{t-1} (1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta})^{t-1} (2t+1) \epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} \\
 & \stackrel{(k)}{\leq} \frac{1+2t}{4^{t-1}} \epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} \\
 & \stackrel{(\ell)}{\leq} 3\epsilon_\dagger \frac{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right]^{-1}}{(1-2q)^4} \\
 & = \frac{3\epsilon_\dagger}{\left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right] (1-2q)^4} \\
 & \leq \eta. \tag{144}
 \end{aligned}$$

Thus, we have

$$\epsilon_\dagger \leq \frac{\eta}{3} \left[e \tanh(\beta) \log \left(\frac{1}{\tanh(\beta)} \right) \right] (1-2q)^4. \tag{145}$$

In (a) The path is considered as $(\mathcal{F}_0, e_1, \mathcal{F}_1, e_1, \dots, \mathcal{F}_{t-1}, e_t, \mathcal{F}_t)$ where \mathcal{F}_i are segments with all strong edges, e_i are the weak edges and we use the notation $\mu_{\mathcal{F}_i}^\dagger \triangleq \prod_{e \in \mathcal{F}_i} \mu_e^\dagger$ and $\hat{\mu}_{\mathcal{F}_i}^\dagger \triangleq \prod_{e \in \mathcal{F}_i} \hat{\mu}_e^\dagger$. In (b) telescoping sum and triangle inequality is used, In (c),(d),(e),(f) we use the inequalities $|\mu_{\mathcal{F}_i}^\dagger| \leq (1-2q)^{2d_i}$, $|\hat{\mu}_{\mathcal{F}_i}^\dagger| \leq (1-2q)^{2d_i}$, $|\mu_{e_i}^\dagger| \leq \tau^\dagger$, $|\hat{\mu}_{e_i}^\dagger| \leq \tau^\dagger + \epsilon_\dagger$ which holds under $\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger)$ and $|\hat{\mu}_{\mathcal{F}_i}^\dagger - \mu_{\mathcal{F}_i}^\dagger| \leq \gamma_\dagger^{(i)}$ again under $\mathbf{E}_\dagger^{\text{corr}}(\epsilon_\dagger)$ and $\gamma_\dagger^{(i)} = d_i(1-2q)^{2(d_i-1)} \tanh^{d_i-1}(\beta)$ and the triangle inequality. In (g) $\frac{\tau^\dagger + \epsilon_\dagger}{(1-2q)^2} \leq 1$, (h) $d_i \tanh^{d_i-1}(\beta) \leq [-e \tanh(\beta) \log(\tanh(\beta))]^{-1}$ for any $d_i, \beta > 0$. (i) $[-e \tanh(\beta) \log(\tanh(\beta))]^{-1} \geq 1$ holds for any $\beta > 0$. (j) holds since $\tau^\dagger \leq 4\epsilon e^\beta \left(1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta}\right)$ from (142) and (k) holds under the assumption

$$\epsilon_\dagger \leq (1-2q)^2 e^{-\beta} \left[20 \left(1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta}\right)\right]^{-1}. \quad (146)$$

Finally, (l) is true for any $\forall t \in \mathbb{N}$. Thus, ϵ_\dagger should be chosen such that

$$\epsilon_\dagger \leq \min \left\{ \frac{e^{-\beta}(1-2q)^2}{20 \left(1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta}\right)}, e \tanh(\beta) \log\left(\frac{1}{\tanh(\beta)}\right) \frac{\eta}{3} (1-2q)^4 \right\}. \quad (147)$$

In a similar way as Lemma 6.1 by Bresler and Karzand (2018) we get $\mathcal{L}^{(2)}(P, \Pi_{\text{TCL}}(P)) \leq \frac{\eta}{2}$ under the assumption

$$\epsilon_\dagger \leq \min \left\{ \frac{\eta}{16} (1-2q)^2, \frac{(1-2q)^2 e^{-\beta}}{24 \left(1 + 2e^\beta \sqrt{2(1-q)q \tanh \beta}\right)} \right\}. \quad (148)$$

The above is true since

$$\mathcal{L}^{(2)}(P, \Pi_{\text{TCL}}(P)) = \frac{1}{2} \left| \prod_{e \in \text{path}_{\text{T}}(w, \tilde{w})} \frac{\mu_e^\dagger}{(1-2q)^2} - \prod_{e \in \text{path}_{\text{TCL}}(w, \tilde{w})} \frac{\mu_e^\dagger}{(1-2q)^2} \right| \quad (149)$$

$$= \frac{1}{2} \left| \prod_{e \in \text{path}_{\text{T}}(w, \tilde{w})} \mu_e - \prod_{e \in \text{path}_{\text{TCL}}(w, \tilde{w})} \mu_e \right|. \quad (150)$$

The proof by Bresler and Karzand (2018, page 16) slightly changes as follows

$$\begin{aligned} 2\mathcal{L}^{(2)}(P, \Pi_{\text{TCL}}(P)) &\leq |\mu_f \mu_A \mu_{\tilde{A}} \mu_B \mu_{\tilde{B}}| |\mu_C^2 \mu_{\tilde{C}}^2 - 1| + |\mu_f| \left(\Delta(k) + \Delta(\tilde{k}) + \Delta(\tilde{k}) \Delta(k) \right) \\ &= \left| \frac{\mu_f^\dagger}{(1-2q)^2} \mu_A \mu_{\tilde{A}} \mu_B \mu_{\tilde{B}} \right| \left| \mu_C^2 \mu_{\tilde{C}}^2 - 1 \right| \\ &\quad + \left| \frac{\mu_f^\dagger}{(1-2q)^2} \right| \left(\Delta(k) + \Delta(\tilde{k}) + \Delta(\tilde{k}) \Delta(k) \right) \end{aligned}$$

$$\leq 8 \frac{\epsilon_{\dagger}}{(1-2q)^2} + \frac{\tau^{\dagger}}{(1-2q)^2} (2\eta + \eta^2), \quad (151)$$

where the last inequality holds since $|\mu_f^{\dagger}| - |\mu_g^{\dagger}| \leq 4\epsilon_{\dagger}$, $|\mu_f| \left(1 - \mu_C^2 \mu_C^2\right) \leq 2|\mu_f| - 2|\mu_g|$, $|\mu_f^{\dagger}| \leq \tau^{\dagger}$. The modification above is necessary, since for our problem we should be able to work with and derive results based only the parameters n_{\dagger} , ϵ_{\dagger} , τ^{\dagger} , γ_{\dagger} , instead of n , ϵ , τ , γ . Notice that $e \tanh(\beta) \log\left(\frac{1}{\tanh(\beta)}\right) \leq 1$ for all $\beta \in \mathbb{R}^+$, as a consequence of (147), and (148) it is sufficient to assume that

$$\epsilon_{\dagger} \leq \min \left\{ e \tanh(\beta) \log\left(\frac{1}{\tanh(\beta)}\right) \frac{\eta}{16} (1-2q)^4, \frac{(1-2q)^2 e^{-\beta}}{24 \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta}\right)} \right\}. \quad (152)$$

For the sake of saving space, we define

$$c_1(\eta, \beta, q) \triangleq \frac{512}{\left[e \tanh(\beta) \log\left(\frac{1}{\tanh(\beta)}\right) \eta (1-2q)^4 \right]^2}, \quad (153)$$

$$c_2(\beta, q) \triangleq \frac{1152e^{2\beta}}{(1-2q)^4} \left(1 + 2e^{\beta} \sqrt{2(1-q)q \tanh \beta}\right)^2. \quad (154)$$

Finally, the sufficient number of samples satisfy the inequality

$$n_{\dagger} > \max \{c_1(\eta, \beta, q), c_2(\beta, q)\} \log\left(\frac{4p^2}{\delta}\right). \quad (155)$$

The last inequality completes the proof. \blacksquare

D.2 Theorem 3.7: Upper bound on the symmetric KL-Divergence with high probability

Assume the Ising model tree distributions P_{θ} according to a tree $T_{\theta} = (\mathcal{V}, \mathcal{E}_{\theta})$ and the estimate $P_{\theta'}$ according a tree $T_{\theta'} = (\mathcal{V}, \mathcal{E}_{\theta'})$ The goal is to upper bound the symmetric KL divergence

$$S_{\text{KL}}(\theta || \theta') = \sum_{s,t \in \mathcal{E}} (\theta_{st} - \theta'_{st}) (\mu_{st} - \mu'_{st})$$

with high probability. Under the event $E^{\text{corr}}(\epsilon)$ we can upper bound the quantity $|\mu_{st} - \mu'_{st}|$ for all $(s, t) \in \mathcal{E}$ with high probability.

By using bounds $|\theta_{st} - \theta'_{st}| \leq 2\beta$ and $|\mu_{st} - \mu'_{st}| \leq \epsilon$ for all $(s, t) \in \mathcal{E}$ under the event $E^{\text{corr}}(\epsilon)$ Bresler and Karzand (2018), we have

$$\begin{aligned} S_{\text{KL}}(\theta || \theta') &= |S_{\text{KL}}(\theta || \theta')| \\ &= \left| \sum_{s,t \in \mathcal{E}} (\theta_{st} - \theta'_{st}) (\mu_{st} - \mu'_{st}) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{s,t \in \mathcal{E}} |\theta_{st} - \theta'_{st}| |\mu_{st} - \mu'_{st}| \\
 &\leq (p-1) |\beta - (-\beta)| \epsilon \\
 &\leq \eta_S,
 \end{aligned} \tag{156}$$

by assuming $\epsilon \leq \frac{\eta_S}{2\beta(p-1)}$. The sufficient number of samples satisfies the inequality

$$n \geq 4 \log(p^2/\delta) \frac{\beta^2(p-1)^2}{\eta_S^2}. \tag{157}$$

Now assume that n_{\dagger} samples of \mathbf{Y} are given, by using the estimate $\mathbf{P}_{\theta'} = \Pi_{\text{TCL}}(\hat{P}^{\dagger})$ defined in (136) under the event $E_{\dagger}^{\text{corr}}(\epsilon_{\dagger})$ we have $\left| \mu_{st} - \frac{\hat{\mu}_{st}^{\dagger}}{(1-2q)^2} \right| \leq \frac{\epsilon_{\dagger}}{(1-2q)^2}$ from Lemma B.1. In the same way by assuming $\epsilon_{\dagger} \leq \frac{\eta_S(1-2q)^2}{2\beta(p-1)}$, we get

$$n_{\dagger} \geq 4 \log(p^2/\delta) \frac{\beta^2(p-1)^2}{(1-2q)^4 \eta_S^2}. \tag{158}$$

Appendix E. Theorem 3.2 and Theorem 3.4: Proofs

We combine Fano's inequality and a Strong Data Processing Inequality to prove the necessary number of samples in the hidden model setting, first for structure learning (Theorem 3.2) and then for inference (Theorem 3.4). We use the following variation of Fano's inequality.

Lemma E.1 (Tsybakov, 2009, Corollary 2.6): *Assume that Θ is a family of $M+1$ distributions $\theta_0, \theta_1, \dots, \theta_M$ such that $M \geq 2$. Let P_{θ_i} be the distribution of the variable X under the model θ_i , if*

$$\frac{1}{M+1} \sum_{i=1}^M \mathbf{D}_{KL}(P_{\theta_i} \| P_{\theta_0}) \leq \gamma \log M, \quad \text{for } \gamma \in (0, \frac{1}{8}) \tag{159}$$

then for the probability of the error p_e the following inequality holds: $p_e \geq \frac{\log(M+1)-1}{\log(M)} - \gamma$. We restrict the values of γ to $(0, \frac{1}{8})$ because we are interested in the case where $p_e \geq \frac{1}{2}$, in general the above holds for all values of $\gamma \in (0, 1)$, see (Tsybakov, 2009, Corollary 2.6).

Lemma E.1, Bresler's and Karzand's 1st method, and the Strong Data Processing Inequality for the BSC give bound of Theorem 3.2. We start by presenting Bresler's and Karzand's method, which gives a sufficiently tight upper bound on symmetric KL divergence.

Bresler's and Karzand's 1st method: Consider a family of $M+1$ different Ising model distributions $\{P_{\theta_i} : i \in \{0, \dots, M\}\}$, this family of the structured distributions is chosen such that the structure recovery task (through The Chow-Liu algorithm) is sufficiently hard. First, we define P_{θ_0} to be an Ising model distribution with underlying structure a chain with p nodes and parameters $\theta_{j,j+1}^0 = \alpha$, when j is odd and $\theta_{j,j+1}^0 = \beta$ when j is even. The rest of family is constructed as follows: the elements of each θ^i , $i \in [M]$ are equal to the elements

of θ^0 apart from two elements $\theta_{j,j+1}^i = 0$ and $\theta_{j,j+2}^i = \alpha$, for each odd value of j . There are $M + 1$ distributions in the constructed family. Through the expression (33), we derive the following upper bound on the $\mathcal{S}_{\text{KL}}(P_{\theta^0}||P_{\theta^i})$, for all $i \in [M]$, (Bresler and Karzand, 2018, Section 7.1),

$$\mathcal{S}_{\text{KL}}(P_{\theta^0}||P_{\theta^i}) \leq 4\alpha \tanh(\alpha)e^{-2\beta}, \forall i \in [M]. \quad (160)$$

Strong Data Processing Inequality: For each distribution P_{θ^i} and $i \in \{0, \dots, M\}$ we consider the distribution of the noisy variable in the hidden model $P_{\theta^i}^\dagger \triangleq P_{\mathbf{Y}|\mathbf{X}} \circ P_{\theta^i}$ and we would like to find an upper bound for the quantities $\mathcal{S}_{\text{KL}}(P_{\theta^0}^\dagger||P_{\theta^i}^\dagger)$. To do this, we use a strong data processing inequality result. Polyanskiy and Wu (2017) recently proved a strong data processing inequality result for the binary symmetric channel. The input random variable \mathbf{X} is considered to have correlated binary elements, while the noise variables N_i are i.i.d Rademacher(q). This scheme is equivalent to the hidden model that we consider in this paper. In fact we have the following bound

$$\eta_{\text{KL}} \leq 1 - (4q(1 - q))^p, \quad (161)$$

which is proved by Polyanski (Polyanskiy and Wu, 2017, "Evaluation for the BSC", equation (39)), where the quantity η_{KL} is defined as

$$\eta_{\text{KL}} \triangleq \sup_Q \sup_{P: 0 < \mathcal{D}_{\text{KL}}(P||Q) < \infty} \frac{\mathcal{D}_{\text{KL}}(P_{\mathbf{Y}|\mathbf{X}} \circ P || P_{\mathbf{Y}|\mathbf{X}} \circ Q)}{\mathcal{D}_{\text{KL}}(P||Q)}, \quad (162)$$

$P_{\mathbf{Y}|\mathbf{X}}$ is the distribution of the BSC and P, Q are any distributions of the input variable \mathbf{X} .

Since the supremum in (162) is with respect to all possible distributions, it covers any pair of distributions in the desired family $\{P_{\theta^j} : j \in \{0, \dots, M\}\}$. Thus, for all $k, \ell \in \{0, 1, \dots, M\}$ and $k \neq \ell$, it is true that

$$\frac{\mathcal{D}_{\text{KL}}(P_{\theta^k}^\dagger || P_{\theta^\ell}^\dagger)}{\mathcal{D}_{\text{KL}}(P_{\theta^k} || P_{\theta^\ell})} \leq 1 - (4q(1 - q))^p, \quad (163)$$

which comes from (161),(162) and implies the following

$$\mathcal{S}_{\text{KL}}(P_{\theta^k}^\dagger || P_{\theta^\ell}^\dagger) \leq [1 - (4q(1 - q))^p] \mathcal{S}_{\text{KL}}(P_{\theta^k} || P_{\theta^\ell}), \quad \forall k \neq \ell \in \{0, 1, \dots, M\}. \quad (164)$$

We combine (160) and (164) to get

$$\mathcal{S}_{\text{KL}}(P_{\theta^k}^\dagger || P_{\theta^i}^\dagger) \leq [1 - (4q(1 - q))^p] 4\alpha^2 e^{-2\beta}. \quad (165)$$

Finally, from (165) and Lemma E.1 we derive the first part of Theorem 3.2.

Bresler's and Karzand's 2nd method: Theorem 3.4 is the extended version of Corollary 6.3 by Bresler and Karzand (2018) to the hidden model. Following a similar technique, we consider chain structured Ising models with parameters θ^j for $j \in [M]$ such that $\theta_{j,j+1}^j = \alpha$ and $\theta_{i,i+1}^j = \arctanh(\tanh(\alpha) + 2\eta)$, for all $i \neq j$. Then

$$\mathcal{L}^{(2)}(P_{\theta^j}, P_{\theta^{j'}}) = \max_{s,t} |\mathbb{E}_{\theta^j}[X_s X_t] - \mathbb{E}_{\theta^{j'}}[X_s X_t]| \geq 2\eta \quad (166)$$

and

$$\mathbf{S}_{\text{KL}}(P_{\theta^j}, P_{\theta^{j'}}) \leq 2\eta [\operatorname{arctanh}(\tanh(\alpha) + 2\eta) - \alpha] \leq 2\eta \frac{2\eta}{1 - [\tanh(\alpha) + 2\eta]^2}, \quad (167)$$

where the last inequality is a consequence of Mean Value Theorem (see (Bresler and Karzand, 2018, Corollary 6.3) for the original statement). We derive the bound of Theorem 3.4 by combining the strong data processing inequality (162) with (161), (167) and Lemma E.1.

Appendix F. Supplementary Analysis

The next lemma studies the extension of Lemma 8.7 by Bresler and Karzand (2018) to the hidden model. We apply the original method on the hidden model to derive a possibly tighter bound on the sample complexity. Through this approach, the final result has an intractable form.

Lemma F.1 *Choose any $\epsilon_q^* \geq 0$ and δ such that $\delta > \epsilon_q^*$, $\epsilon_q^* = 0$ if and only if $q = 0$. Suppose*

$$n > \max \left\{ \frac{[30y^2(q, n, p, \beta)][4 + G(q, n, p, \beta)/2]}{7(\gamma - \Delta(q, n, p, \beta))^2} \log \frac{2p^2}{(\delta - \epsilon_q^*)(\frac{1}{2} + \epsilon_q^*)}, \quad (168)$$

$$108 \frac{1}{1 - (1 - 2q)^4 \tanh^2(\beta)} \log(2p^2/\delta) \right\} \quad (169)$$

and

$$\frac{\gamma - \Delta(q, n, p, \beta)}{y(q, n, p, \beta)G(q, n, p, \beta)} \in (0, 0.4), \quad (170)$$

where

$$\Delta(q, n, p, \beta) \triangleq \frac{F_{\epsilon_q^*, p}(q, n)}{(1 - 2q)^2} [-e \tanh(\beta) \log(\tanh(\beta))]^{-1}, \quad (171)$$

$$y(q, n, p, \beta) \triangleq 2 + F_{\epsilon_q^*(q, n)} \left(\frac{\tanh(\beta)}{1 - 2q} \right)^{2(p-1)}, \quad (172)$$

$$G(q, n, p, \beta) \triangleq \sup_{d \in [p-1]} \sum_{k=1}^d \frac{1 - (\mu_k^\dagger)^2}{(1 - 2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1 - 2q)^2} \right)^2 + 2 \frac{\gamma_j^\dagger}{(1 - 2q)^2} \right] \quad (173)$$

$$+ \left[\frac{2F_{\epsilon_q^*, p}(q, n) + (F_{\epsilon_q^*, p}(q, n))^2}{(1 - 2q)^4} \right] [-e \tanh^2(\beta) \log(\tanh^2(\beta))]^{-1}, \quad (174)$$

and where for the function $F_{\epsilon_q^*, p}$, it is true that, if $q = 0$ then $F_{\epsilon_q^*, p}(q, n) = 0$ for every $n \geq 0$, and $\lim_{n \rightarrow \infty} F_{\epsilon_q^*, p}(q, n) = 0$ for every $q \neq 0$. Then, it is true that

$$\mathbb{P} \left[\left| \prod_{i=1}^d \frac{\hat{\mu}_i^\dagger}{(1 - 2q)^2} - \prod_{i=1}^d \frac{\mu_i^\dagger}{(1 - 2q)^2} \right| > \gamma \right] \leq 2\delta/p^2, \quad \forall d \geq 2. \quad (175)$$

Proof Define the event $E_{e,\dagger}^{\text{edge}}$ as

$$E_{e,\dagger}^{\text{edge}} \triangleq \left\{ \left| \frac{\hat{\mu}_e^\dagger}{(1-2q)^2} - \frac{\mu_e^\dagger}{(1-2q)^2} \right| \leq \frac{\gamma_e}{(1-2q)^2} \right\}, \quad e \in \mathcal{E}_T, \quad (176)$$

and a fixed $q \in [0, 1/2)$. The variance of $\hat{\mu}_e^\dagger/(1-2q)^2$ is $(1 - (\mu_e^\dagger)^2)/n(1-2q)^4$ and by applying Bernstein's inequality

$$\mathbb{P} \left[\left(E_{e,\dagger}^{\text{edge}} \right)^c \right] \leq 2 \exp \left(- \frac{n\gamma_e^2/(1-2q)^4}{2 \frac{1 - (\mu_e^\dagger)^2}{(1-2q)^4} + \frac{4}{3} \gamma_e/(1-2q)^2} \right), \quad \forall \gamma_e > 0. \quad (177)$$

Under the assumption $\gamma_e = \sqrt{3 \frac{1 - (\mu_e^\dagger)^2}{n} \log(2p/\delta)}$, if $n > 108 \frac{1}{1 - (1-2q)^4 \tanh^2(\beta)} \log(2p^2/\delta)$ then $\mathbb{P} \left[\left(E_{\dagger}^{\text{edge}}(\mathcal{E}_T) \right)^c \right] \leq \delta$, similarly to Lemma E.1 by Bresler and Karzand (2018).

Define the random variable

$$M_i^\dagger \triangleq \left(\frac{\hat{\mu}_i^\dagger}{(1-2q)^2} - \frac{\mu_i^\dagger}{(1-2q)^2} \right) \prod_{j=1}^{i-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=i+1}^d \frac{\mu_j^\dagger}{(1-2q)^2}. \quad (178)$$

Then $\sum_{i=1}^d M_i^\dagger = \prod_{i=1}^d \frac{\hat{\mu}_i^\dagger}{(1-2q)^2} - \prod_{i=1}^d \frac{\mu_i^\dagger}{(1-2q)^2}$, $\mathcal{A}_k \triangleq \{e_1, e_2, \dots, e_k\}$, for $2 \leq k \leq d$.

Let E_q be an event such that if $q=0$ then $E_q^c \equiv \emptyset$, $\mathbb{P}(E_q^c) \leq \epsilon_q^*$ for some $\epsilon_q^* \geq 0$. The event E_q will be later used to apply Egorov's Theorem, at this point, it can be any event that satisfies the above conditions. The law of total probability gives

$$\begin{aligned} \mathbb{P} \left[\left| \sum_{i=1}^d M_i^\dagger \right| > \gamma \right] &\leq \mathbb{P} \left[\left| \sum_{i=1}^d M_i^\dagger \right| > \gamma \mid E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right] + \mathbb{P} \left[\left(E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right)^c \right] \\ &\leq \mathbb{P} \left[\left| \sum_{i=1}^d M_i^\dagger \right| > \gamma \mid E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right] + \mathbb{P} \left[\left(E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right)^c \right] + \mathbb{P}[(E_q)^c] \\ &\leq \mathbb{P} \left[\left| \sum_{i=1}^d M_i^\dagger \right| > \gamma \mid E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right] + \mathbb{P} \left[\left(E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right)^c \right] + \epsilon_q^*. \end{aligned} \quad (179)$$

For any $k \leq d$,

$$\begin{aligned} &\mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^k M_i^\dagger \right) \mid E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{k-1}) \right] \\ &= \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \mathbb{E}_q \left[\exp \left(\lambda M_k^\dagger \right) \mid \hat{\mu}_1^\dagger, \dots, \hat{\mu}_{k-1}^\dagger \right] \right], \end{aligned} \quad (180)$$

where the integration \mathbb{E}_q is with respect to the measure $\mathbb{P}_{M_1^\dagger, \dots, M_k^\dagger} \left[(\cdot) \mid E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{d-1}) \right]$.

$$M_k^\dagger = \left(\frac{\frac{1}{n} \sum_{\ell=1}^n (X_k N_k X_{k+1} N_{k+1})^{(\ell)}}{(1-2q)^2} - \frac{\mu_k^\dagger}{(1-2q)^2} \right) \prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2}$$

$$= \frac{1}{n} \sum_{\ell=1}^n \left(\frac{(X_k N_k X_{k+1} N_{k+1})^{(\ell)}}{(1-2q)^2} - \frac{\mu_i^\dagger}{(1-2q)^2} \right) \prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2}. \quad (181)$$

Define

$$Z_k^{(\ell)} \triangleq \left(\frac{(X_k N_k X_{k+1} N_{k+1})^{(\ell)}}{(1-2q)^2} - \frac{\mu_i^\dagger}{(1-2q)^2} \right) \prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2}. \quad (182)$$

The random variables $Z_k^{(\ell)}$ are not independent conditioned on the event $E_q \cap E_\dagger^{\text{edge}}(\mathcal{A}_{d-1})$. To apply a concentration of measure result on $Z_k^{(\ell)}$ we use the extended Bennet's inequality for supermartingales Fan et al. (2012).

Martingale Differences: Define $\xi_k^{(0)} \triangleq 0$, $\xi_k^{(1)} \triangleq Z_k^{(1)} - \mathbb{E}_q [Z_k^{(1)} | \hat{\mu}_{k-1}^\dagger, \dots, \hat{\mu}_1^\dagger]$, $\xi_k^{(i)} \triangleq Z_k^{(i)} - \mathbb{E}_q [Z_k^{(i)} | Z_k^{(i-1)}, \dots, Z_k^{(1)}, \hat{\mu}_{k-1}^\dagger, \dots, \hat{\mu}_1^\dagger]$. Also, define as \mathcal{F}_{i-1}^k the σ -algebra generated by $Z_k^{(i-1)}, \dots, Z_k^{(1)}, \hat{\mu}_{k-1}^\dagger, \dots, \hat{\mu}_1^\dagger$, then $(\xi_k^{(i)}, \mathcal{F}_i^k)_{i=1, \dots, n}$ is a Martingale Difference Sequence (MDS). Additionally, conditioned on $Z_k^{(i-1)}, \dots, Z_k^{(1)}, \hat{\mu}_{k-1}^\dagger, \dots, \hat{\mu}_1^\dagger$ we have

$$Z_k^{(i)} = \begin{cases} \frac{1}{(1-2q)^{2d}} (1 - \mu_k^\dagger) \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger, & \text{with probability } \frac{1 + \mu_k^\dagger + f_{q,n,k}(\omega)}{2} \\ -\frac{1}{(1-2q)^{2d}} (1 + \mu_k^\dagger) \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger, & \text{with probability } \frac{1 - \mu_k^\dagger - f_{q,n,k}(\omega)}{2}. \end{cases} \quad (183)$$

Thus

$$\mathbb{E}_q [Z_k^{(i)} | \mathcal{F}_{i-1}^k] = \mathbb{E}_q [Z_k^{(i)} | Z_k^{(i-1)}, \dots, Z_k^{(1)}, \hat{\mu}_{k-1}^\dagger, \dots, \hat{\mu}_1^\dagger] = \frac{f_{q,n,k}(\omega)}{(1-2q)^{2d}} \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger, \quad (184)$$

and

$$\mathbb{E}_q \left[\left(\xi_k^{(i)} \right)^2 \middle| \mathcal{F}_{i-1}^k \right] = \mathbb{E}_q \left[\left(Z_k^{(i)} - \mathbb{E}_q [Z_k^{(i)} | \mathcal{F}_{i-1}^k] \right)^2 \middle| \mathcal{F}_{i-1}^k \right] \quad (185)$$

$$= \frac{1 - \left(\mu_k^\dagger + f_{q,n,k}(\omega) \right)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2. \quad (186)$$

The function $f_{q,n,k}(\cdot)$ is equal to zero when $q = 0$, which corresponds to the noiseless setting. The limits $\lim_{q \rightarrow 0} f_{q,n,k}(\omega) \xrightarrow{a.s.} 0$ and $\lim_{n \rightarrow \infty} f_{q,n,k}(\omega) \xrightarrow{a.s.} 0$ come from the almost sure convergence of the conditional distribution $\mathbb{P} \left(Z_k^{(i)} | \mathcal{F}_{i-1}^k \right)$, under the sense that for any sequence of n which goes to infinity, it is true that

$$\lim_{n \rightarrow \infty} f_{q,n,k}(\omega) \xrightarrow{a.s.} 0, \quad \forall q \in [0, 1/2) \quad (187)$$

Furthermore, for any sequence q_1, q_2, \dots, q_m , such that $\lim_{m \rightarrow \infty} q_m \rightarrow 0$,

$$\lim_{m \rightarrow \infty} f_{q_m, n}(\mathcal{F}_{i-1}^k) \xrightarrow{a.s.} 0, \quad \forall n \in \mathbb{N}. \quad (188)$$

The set E_q , previously introduced, is necessary for the existence of a uniform upper bound on the measurable function $|f_{q,n,k}(\cdot)|$. A uniform upper bound on $|f_{q,n,k}(\cdot)|$, such that it converges to zero when $n \rightarrow \infty$ or $q_m \rightarrow 0$, does not exist in general. The last can be proved by finding a sequence of sets in the filtration \mathcal{F}_{i-1}^k which give an upper bound invariant of n . However, we know that the almost sure convergence holds because of (187) and (188), and we can apply Egorov's Theorem.

Theorem F.2 Egorov's Theorem (see Bogachev (2007)): *Let f_n be a sequence of M -valued measurable functions, where M is a separable metric space, on some measure space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose there is a measurable subset $A \subset \Omega$, with finite \mathbb{P} -measure, such that f_n converges \mathbb{P} -almost everywhere on A to a limit function f . Then, for every $\epsilon > 0$, there exists a measurable subset B of A such that $\mathbb{P}(B) < \epsilon$, such that f_n converges to f uniformly on the relative complement $A \setminus B$. This type of convergence is called **almost uniform**.*

We choose the event E_q such that under E_q the function $|f_{q,n,k}(\cdot)|$ converges uniformly and $\mathbb{P}(E_q) \geq 1 - \epsilon_q^*$. That is, for fixed q and $\epsilon_q^* > 0$ there exist $N \in \mathbb{N}$ such that for all $n > N$

$$|f_{q,n,k}(\omega)| \leq F_{\epsilon_q^*, p}(q, n), \quad (189)$$

and

$$\lim_{n \rightarrow \infty} F_{\epsilon_q^*, p}(q, n) \rightarrow 0. \quad (190)$$

The uniform convergence gives

$$\left| \mathbb{E}_q \left[Z_k^{(i)} | \mathcal{F}_{i-1}^k \right] \right| \leq \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \quad (191)$$

and

$$\begin{aligned} & \left| \mathbb{E}_q \left[\left(\xi_k^{(i)} \right)^2 | \mathcal{F}_{i-1}^k \right] \right| \\ &= \frac{1 - \left(\mu_k^\dagger + f_{q,n,k}(\omega) \right)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 \\ &\leq \frac{1 - \left(|\mu_k^\dagger| - |f_{q,n,k}(\omega)| \right)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 \\ &\leq \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 + \frac{2|\mu_k^\dagger| |f_{q,n,k}(\omega)| + (f_{q,n,k}(\omega))^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 \\ &\leq \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 + \frac{2|\mu_k^\dagger| F_{\epsilon_q^*, p}(q, n) + \left(F_{\epsilon_q^*, p}(q, n) \right)^2}{(1-2q)^{4d}} \left[\prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right]^2 \end{aligned}$$

$$\begin{aligned}
 &= \left[\prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2} \right]^2 \\
 &\quad \times \left[\frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} + \frac{2|\mu_k^\dagger|F_{\epsilon_q^*,p}(q,n) + (F_{\epsilon_q^*,p}(q,n))^2}{(1-2q)^4} \right]. \tag{192}
 \end{aligned}$$

Under the event $E_{\dagger}^{\text{edge}}(\mathcal{A}_{k-1})$, we have

$$\left(\frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \right)^2 \leq \left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2\frac{\gamma_j^\dagger}{(1-2q)^2} \tag{193}$$

and we get

$$\begin{aligned}
 &\left| \mathbb{E}_q \left[\left(\xi_k^{(i)} \right)^2 \middle| \mathcal{F}_{i-1}^k \right] \right| \\
 &\leq \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2\frac{\gamma_j^\dagger}{(1-2q)^2} \right] \\
 &\quad + \left[\prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2} \right]^2 \left[\frac{2|\mu_k^\dagger|F_{\epsilon_q^*,p}(q,n) + (F_{\epsilon_q^*,p}(q,n))^2}{(1-2q)^4} \right] \\
 &\triangleq g_k(q, n). \tag{194}
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 |\xi_k^{(i)}| &\leq 2 + \left| \mathbb{E}_q \left[Z_k^{(i)} \middle| \mathcal{F}_{i-1}^k \right] \right| \leq 2 + \frac{F_{\epsilon_q^*(q,n)}}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \\
 &\leq 2 + F_{\epsilon_q^*(q,n)} \left(\frac{\tanh(\beta)}{1-2q} \right)^{2(p-1)} \triangleq y(q, n, p, \beta).
 \end{aligned}$$

Then for any $k \leq d$,

$$\begin{aligned}
 &\mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^k M_i^\dagger \right) \middle| E_q \cap E_{\dagger}^{\text{edge}}(\mathcal{A}_{k-1}) \right] \\
 &= \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \mathbb{E}_q \left[\exp \left(\lambda M_k^\dagger \right) \middle| \hat{\mu}_1^\dagger, \dots, \hat{\mu}_{k-1}^\dagger \right] \right] \tag{195}
 \end{aligned}$$

$$= \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \mathbb{E}_q \left[\exp \left(\lambda \frac{1}{n} \sum_{i=1}^n Z_k^{(i)} \right) \middle| \hat{\mu}_1^\dagger, \dots, \hat{\mu}_{k-1}^\dagger \right] \right] \tag{196}$$

$$\begin{aligned}
 &= \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \right. \\
 &\quad \times \mathbb{E}_q \left[\exp \left(\lambda \frac{1}{n} \sum_{i=1}^n \xi_k^{(i)} \right) \exp \left(\lambda \frac{1}{n} \sum_{i=1}^n \mathbb{E}_q \left[Z_k^{(i)} | \mathcal{F}_{i-1}^k \right] \right) \middle| \hat{\mu}_1^\dagger, \dots, \hat{\mu}_{k-1}^\dagger \right] \quad (197)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \exp \left(\lambda \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \right) \\
 &\quad \times \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \mathbb{E}_q \left[\exp \left(\lambda \frac{1}{n} \sum_{i=1}^n \xi_k^{(i)} \right) \middle| \hat{\mu}_1^\dagger, \dots, \hat{\mu}_{k-1}^\dagger \right] \right] \quad (198)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \exp \left(\lambda \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \right) \\
 &\quad \times \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \exp \left\{ n F \left(\frac{\mathbb{E}_q \left[\left(\xi_k^{(i)} \right)^2 \middle| \mathcal{F}_{i-1}^k \right]}{y^2(q, n)}, \left| \lambda \frac{y(q, n)}{n} \right| \right) \right\} \right] \quad (199)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \exp \left(\lambda \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \right) \\
 &\quad \times \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \exp \left\{ n F \left(\frac{g_d(q, n)}{4}, \left| \lambda \frac{y(q, n)}{n} \right| \right) \right\} \right] \quad (200)
 \end{aligned}$$

$$\begin{aligned}
 &= \exp \left(\lambda \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \right) \\
 &\quad \times \exp \left\{ n F \left(\frac{g_d(q, n)}{4}, \left| \lambda \frac{y(q, n)}{n} \right| \right) \right\} \mathbb{E}_q \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \right] \\
 &= \exp \left(\lambda \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \left| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right| \right) \exp \left\{ n F \left(\frac{g_d(q, n)}{4}, \left| \lambda \frac{y(q, n)}{n} \right| \right) \right\} \\
 &\quad \times \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right] \quad (201)
 \end{aligned}$$

(195) comes from change of measure and tower property, the definition of M_k^\dagger and $\xi_k^{(i)}$ give (196) and (197). The uniform upper bound in (191) gives (198). (199) is the upper bound on the moment generating function of the supermartingale Fan et al. (2012). (200) comes from the uniform upper bound of the second order moment (194). To get a recurrence

$$\begin{aligned}
 & \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right] \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right) \\
 &= \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \cap \mathbb{E}_{e_{k-1}, \dagger}^{\text{edge}} \right] \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right) \\
 &= \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \cap \mathbb{E}_{e_{k-1}, \dagger}^{\text{edge}} \right] \\
 &\quad \times \mathbb{P} \left(\mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right) \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right) \\
 &= \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \cap \mathbb{E}_{e_{k-1}, \dagger}^{\text{edge}} \right] \\
 &\quad \times \mathbb{P} \left(\mathbb{E}_{e_{k-1}, \dagger}^{\text{edge}} \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right) \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right) \\
 &= \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \mathbf{1}_{\mathbb{E}_{e_{k-1}, \dagger}^{\text{edge}}} \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right] \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right) \\
 &\leq \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^{k-1} M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right] \mathbb{P} \left(\mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-2}) \right). \tag{202}
 \end{aligned}$$

By applying the recurrence, we derive the following bound

$$\begin{aligned}
 & \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^k M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right] \\
 &\leq \exp \left(\lambda \frac{dF_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \middle| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right) \exp \left\{ n \sum_{k=1}^d F \left(\frac{g_k(q, n)}{4}, |\lambda| \frac{y(q, n)}{n} \right) \right\} \\
 &\leq \exp \left(\lambda \frac{dF_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \middle| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \right) \exp \left\{ nd F \left(\frac{1}{d} \sum_{k=1}^d \frac{g_k(q, n)}{4}, |\lambda| \frac{y(q, n)}{n} \right) \right\}. \tag{203}
 \end{aligned}$$

Next, we find an upper bound of

$$\frac{dF_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \middle| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \middle| \tag{204}$$

as

$$\frac{dF_{\epsilon_q^*, p}(q, n)}{(1-2q)^{2d}} \middle| \prod_{j=1}^{k-1} \hat{\mu}_j^\dagger \prod_{j=k+1}^d \mu_j^\dagger \middle| \leq \frac{dF_{\epsilon_q^*, p}(q, n)}{(1-2q)^2} \tanh^{d-1}(\beta)$$

$$\leq \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^2} [-e \tanh(\beta) \log(\tanh(\beta))]^{-1}, \quad \forall d > 2$$

and we define

$$\Delta(q, n, p, \beta) \triangleq \frac{F_{\epsilon_q^*, p}(q, n)}{(1-2q)^2} [-e \tanh(\beta) \log(\tanh(\beta))]^{-1}. \quad (205)$$

We know that $\lim_{n \rightarrow \infty} \Delta(q, n, p, \beta) = 0$ and $\Delta(0, n, p, \beta) = 0$. Furthermore,

$$\begin{aligned} & \sum_{k=1}^d g_k(q, n) \\ &= \sum_{k=1}^d \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2 \frac{\gamma_j^\dagger}{(1-2q)^2} \right] \\ & \quad + \sum_{k=1}^d \left[\prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2} \right]^2 \left[\frac{2|\mu_k^\dagger| F_{\epsilon_q^*, p}(q, n) + \left(F_{\epsilon_q^*, p}(q, n) \right)^2}{(1-2q)^4} \right] \\ & \leq \sum_{k=1}^d \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2 \frac{\gamma_j^\dagger}{(1-2q)^2} \right] \\ & \quad + d \left[\prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2} \right]^2 \left[\frac{2F_{\epsilon_q^*, p}(q, n) + \left(F_{\epsilon_q^*, p}(q, n) \right)^2}{(1-2q)^4} \right]. \end{aligned} \quad (206)$$

To maximize the term $\sum_{k=1}^d g_k(q, n)$ with respect to d

$$\begin{aligned} & \sup_{d \in [p-1]} \sum_{k=1}^d g_k(q, n) \\ & \leq \sup_{d \in [p-1]} \sum_{k=1}^d \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2 \frac{\gamma_j^\dagger}{(1-2q)^2} \right] \\ & \quad + \sup_{d \in [p-1]} d \left[\prod_{j=1}^{k-1} \frac{\hat{\mu}_j^\dagger}{(1-2q)^2} \prod_{j=k+1}^d \frac{\mu_j^\dagger}{(1-2q)^2} \right]^2 \left[\frac{2F_{\epsilon_q^*, p}(q, n) + \left(F_{\epsilon_q^*, p}(q, n) \right)^2}{(1-2q)^4} \right] \\ & \leq \sup_{d \in [p-1]} \sum_{k=1}^d \frac{1 - (\mu_k^\dagger)^2}{(1-2q)^4} \prod_{j=1, j \neq k}^d \left[\left(\frac{\mu_j^\dagger}{(1-2q)^2} \right)^2 + 2 \frac{\gamma_j^\dagger}{(1-2q)^2} \right] \\ & \quad + \left[\frac{2F_{\epsilon_q^*, p}(q, n) + \left(F_{\epsilon_q^*, p}(q, n) \right)^2}{(1-2q)^4} \right] [-e \tanh^2(\beta) \log(\tanh^2(\beta))]^{-1} \\ & \triangleq G(q, n, p, \beta), \end{aligned} \quad (207)$$

and we know that $G(0, n, p, \beta) = \frac{3}{4}$.

$$\begin{aligned}
 & \mathbb{E} \left[\exp \left(\lambda \sum_{i=1}^k M_i^\dagger \right) \middle| \mathbb{E}_q \cap \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{k-1}) \right] \\
 & \leq \exp(\lambda \Delta(q, n, p, \beta)) \exp \left\{ nd F \left(\frac{1}{d} \frac{G(q, n, p, \beta)}{4}, |\lambda| \frac{y(q, n, p, \beta)}{n} \right) \right\} \\
 & = \exp \left\{ nd F \left(\frac{1}{d} \frac{G(q, n, p, \beta)}{4}, |\lambda| \frac{y(q, n, p, \beta)}{n} \right) + \lambda \Delta(q, n, p, \beta) \right\} \\
 & = \exp \left\{ nd F \left(\frac{G}{4d}, |\lambda| \frac{y}{n} \right) + \lambda \Delta \right\}. \tag{208}
 \end{aligned}$$

For shake of space, denote the functions $G(q, n, p, \beta)$, $y(q, n, p, \beta)$, $\Delta(q, n, p, \beta)$ as G , y , Δ respectively. It is true that

$$\begin{aligned}
 & \mathbb{E} \left[\exp \left(\lambda \sum_{k=1}^d M_k^\dagger \right) \middle| \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \cap \mathbb{E}_q \right] \mathbb{P} \left[\mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \cap \mathbb{E}_q \right] \\
 & \leq \exp \left\{ nd F \left(\frac{G}{4d}, |\lambda| \frac{y}{n} \right) + \lambda \Delta \right\}, \tag{209}
 \end{aligned}$$

where

$$F(t, \lambda) = \log \left(\frac{1}{1+t} e^{-\lambda t} + \frac{t}{1+t} e^\lambda \right). \tag{210}$$

Under the assumption $n > \frac{108}{1-(1-2q)^2 \tanh(\beta)} \log(4p)$, we have $\mathbb{P} \left[\left(\mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \cap \mathbb{E}_q \right)^c \right] \leq \mathbb{P} \left[\left(\mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \cap \mathbb{E}_q \right)^c \right] \leq 1/2 + \epsilon_q^*$. The last inequality gives

$$\mathbb{E}_q \left[\exp \left(\lambda \sum_{k=1}^d M_k^\dagger \right) \middle| \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \right] \leq \frac{1}{\frac{1}{2} + \epsilon_q^*} \exp \left\{ nd F \left(\frac{G}{4d}, |\lambda| \frac{y}{n} \right) + \lambda \Delta \right\}, \tag{211}$$

which implies that

$$\begin{aligned}
 \mathbb{P} \left[\sum_{k=1}^d M_k^\dagger \geq \gamma \middle| \mathbb{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \right] & \leq \frac{1}{\frac{1}{2} + \epsilon_q^*} \min_{\lambda > 0} \exp \left\{ nd F \left(\frac{G}{4d}, \lambda \frac{y}{n} \right) + \lambda \Delta - \lambda \gamma \right\} \\
 & = \frac{1}{\frac{1}{2} + \epsilon_q^*} \min_{\lambda > 0} \exp \left\{ nd F \left(\frac{G}{4d}, \lambda \frac{y}{n} \right) + \lambda (\Delta - \gamma) \right\} \\
 & = \frac{1}{\frac{1}{2} + \epsilon_q^*} \min_{\lambda > 0} \exp \left\{ nd F \left(\frac{G}{4d}, \lambda \frac{y}{n} \right) - \lambda \gamma' \right\}, \tag{212}
 \end{aligned}$$

where $\gamma' \triangleq \gamma - \Delta$. The minimum value is at

$$\lambda^* = \frac{n/y}{1 + G/4d} \log \frac{1 + \frac{4\gamma'}{Gy}}{1 - \frac{\gamma'}{dy}}, \tag{213}$$

by substituting the optimal value we get

$$\begin{aligned}
 & \exp \left\{ nd F \left(\frac{G}{4d}, \lambda^* \frac{y}{n} \right) - \lambda^* \gamma' \right\} \\
 &= \left[\frac{1}{1 + \frac{G}{4d}} \left(\frac{1 + \frac{4\gamma'}{Gy}}{1 - \frac{\gamma'}{dy}} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} + \frac{\frac{G}{4d}}{1 + \frac{G}{4d}} \left(\frac{1 + \frac{4\gamma'}{Gy}}{1 - \frac{\gamma'}{dy}} \right)^{\frac{1}{1+G/4d} \left(1 - \frac{\gamma'}{yd} \right)} \right]^{nd} \\
 &= \left[\frac{1}{1 + \frac{G}{4d}} \left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{G/4d}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)}} \right. \\
 &\quad \left. + \frac{\frac{G}{4d}}{1 + \frac{G}{4d}} \left(1 + \frac{4\gamma'}{Gy} \right)^{\frac{1}{1+G/4d} \left(1 - \frac{\gamma'}{yd} \right)} \left(1 - \frac{\gamma'}{dy} \right)^{-\frac{1}{1+G/4d} \left(1 - \frac{\gamma'}{yd} \right)} \right]^{nd} \\
 &= \left[\frac{1}{1 + \frac{G}{4d}} \left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{G/4d}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)} - 1} \left(1 - \frac{\gamma'}{dy} \right) \right. \\
 &\quad \left. + \frac{\frac{G}{4d}}{1 + \frac{G}{4d}} \left(1 + \frac{4\gamma'}{Gy} \right)^{\frac{1}{1+G/4d} \left(1 - \frac{\gamma'}{yd} \right) - 1} \left(1 + \frac{4\gamma'}{Gy} \right) \left(1 - \frac{\gamma'}{dy} \right)^{-\frac{1}{1+G/4d} \left(1 - \frac{\gamma'}{yd} \right)} \right]^{nd} \\
 &= \left[\frac{1}{1 + \frac{G}{4d}} \left(1 - \frac{\gamma'}{dy} \right) + \frac{\frac{G}{4d}}{1 + \frac{G}{4d}} \left(1 + \frac{4\gamma'}{Gy} \right) \right]^{nd} \\
 &\quad \times \left[\left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{G/4d}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)} - 1} \right]^{nd} \\
 &= 1^{nd} \times \left[\left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{G/4d}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)} - 1} \right]^{nd} \\
 &= \left[\left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{-1}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)}} \right]^{nd}, \tag{214}
 \end{aligned}$$

which gives

$$\begin{aligned}
 & \mathbb{P} \left[\sum_{k=1}^d M_k^\dagger \geq \gamma \mid \mathbf{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \right] \\
 & \leq \frac{1}{\frac{1}{2} + \epsilon_q^*} \left[\left(1 + \frac{4\gamma'}{Gy} \right)^{-\frac{G/4d}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)}} \left(1 - \frac{\gamma'}{dy} \right)^{\frac{-1}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)}} \right]^{nd}. \tag{215}
 \end{aligned}$$

To express the upper bound as an exponential function of γ

$$d \left[\left(\frac{G/4d}{1 + G/4d} + \frac{\gamma'/(dy)}{(1 + G/4d)} \right) \log \left(1 + \frac{4\gamma'}{Gy} \right) + \left(\frac{1}{1 + G/4d} - \frac{\gamma'/(dy)}{(1 + G/4d)} \right) \log \left(1 - \frac{\gamma'}{dy} \right) \right]$$

$$\begin{aligned}
 &\geq d \left[\left(\frac{G/4d}{1+G/4d} + \frac{\gamma'/(dy)}{(1+G/4d)} \right) \left(\frac{4\gamma'}{Gy} - 8 \left[\frac{\gamma'}{Gy} \right]^2 \right) \right. \\
 &\quad \left. - \left(\frac{1}{1+G/4d} - \frac{\gamma'/(dy)}{(1+G/4d)} \right) \log \left(1 + \frac{\gamma'/dy}{1-\gamma'/dy} \right) \right] \\
 &\geq \left[\left(\frac{dG/4}{d+G/4} + \frac{d\gamma'/y}{(d+G/4)} \right) \left(\frac{4\gamma'}{Gy} - 8 \left[\frac{\gamma'}{Gy} \right]^2 \right) \right. \\
 &\quad \left. - \left(\frac{d^2}{d+G/4} - \frac{d\gamma'/y}{(d+G/4)} \right) \left(\frac{\gamma'/dy}{1-\gamma'/dy} - \frac{1}{2} \left(\frac{\gamma'/dy}{1-\gamma'/dy} \right)^2 \right) \right] \\
 &= \frac{d}{d+G/4} \left[(G/4 + \gamma'/y) \left(\frac{4\gamma'}{Gy} - 8 \left[\frac{\gamma'}{Gy} \right]^2 \right) \right. \\
 &\quad \left. - (d - \gamma'/y) \left(\frac{\gamma'/dy}{1-\gamma'/dy} - \frac{1}{2} \left(\frac{\gamma'/dy}{1-\gamma'/dy} \right)^2 \right) \right] \\
 &\geq \frac{d}{d+G/4} \left[(G/4 + \gamma'/y) \left(\frac{4\gamma'}{Gy} - 8 \left[\frac{\gamma'}{Gy} \right]^2 \right) - \gamma'/2y \right] \\
 &= \frac{dG/4}{d+G/4} \left[(1 + 4\gamma'/(yG)) \left(\frac{4\gamma'}{Gy} - 8 \left[\frac{\gamma'}{Gy} \right]^2 \right) - 2\gamma'/(yG) \right] \\
 &= \frac{dG/4}{d+G/4} [(1 + 4\zeta)(4\zeta - 8\zeta^2) - 2\zeta] \\
 &\geq \frac{G/2}{2+G/4} [(1 + 4\zeta)(4\zeta - 8\zeta^2) - 2\zeta] \geq \frac{7\zeta^2}{30} \frac{G/2}{2+G/4}, \quad \forall \zeta \in (0, 0.4), \tag{216}
 \end{aligned}$$

and $\zeta \triangleq \gamma'/(yG)$. The last inequality gives

$$\begin{aligned}
 &\mathbb{P} \left[\sum_{k=1}^d M_k^\dagger \geq \gamma \mid \mathbf{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \right] \\
 &\leq \frac{1}{\frac{1}{2} + \epsilon_q^*} \exp \left(- \frac{7(\gamma - \Delta(q, n, p, \beta))^2}{30y^2(q, n, p, \beta)} \frac{n}{4 + G(q, n, p, \beta)/2} \right). \tag{217}
 \end{aligned}$$

In a similar way we derive the bound for $\mathbb{P} \left[\sum_{k=1}^d M_k^\dagger \leq -\gamma \mid \mathbf{E}_\dagger^{\text{edge}}(\mathcal{A}_{d-1}) \right]$.

Finally, choose any $\epsilon_q^* > 0$ and δ such that $\delta > \epsilon_q^*$. if

$$\begin{aligned}
 n > \max \left\{ \frac{[30y^2(q, n, p, \beta)][4 + G(q, n, p, \beta)/2]}{7(\gamma - \Delta(q, n, p, \beta))^2} \log \frac{2p^2}{(\delta - \epsilon_q^*)(\frac{1}{2} + \epsilon_q^*)}, \right. \\
 &\quad \left. 108 \frac{1}{1 - (1 - 2q)^4 \tanh^2(\beta)} \log(2p^2/\delta) \right\} \tag{218}
 \end{aligned}$$

and

$$\frac{\gamma - \Delta(q, n, p, \beta)}{y(q, n, p, \beta)G(q, n, p, \beta)} \in (0, 0.4), \tag{219}$$

then

$$\mathbb{P} \left[\left| \prod_{i=1}^d \frac{\hat{\mu}_i^\dagger}{(1-2q)^2} - \prod_{i=1}^d \frac{\mu_i^\dagger}{(1-2q)^2} \right| > \gamma \right] \leq 2\delta/p^2, \quad \forall d \geq 2. \quad (220)$$

We found an upper bound of the event $\left\{ \left| \prod_{i=1}^d \frac{\hat{\mu}_i^\dagger}{(1-2q)^2} - \prod_{i=1}^d \frac{\mu_i^\dagger}{(1-2q)^2} \right| > \gamma \right\}$ by using the technique of the proof by Fan et al. (2012), Corollary 2.3 by Fan et.al (2012). We started from the moment generating function conditioned on the intersection of two necessary events $E_q, E_{\dagger}^{\text{edge}}(\mathcal{A}_{k-1})$. For completeness we give the corresponding "unconditioned" concentration of measure bound for our setting. \blacksquare

For the corresponding concentration inequality see also Corollary 2.3 by Fan et.al (2012).

A Structure-Preserving Case: Lemma F.3 considers a special case of tree structures for the hidden variables, the set of edges is a set with disconnected edges, no edge is connected to any other. Then we show that the same structure is preserved for the observable variables.

Lemma F.3 *Assume a graph $G = (\mathcal{V}, \mathcal{E}_G)$, with $|\mathcal{V}| = p$, $|\mathcal{E}_G| = \frac{p}{2} \in \mathbb{N}$ has no edge connected to any other edge. We also assume that $X_i \in \{-1, +1\}$, $\mathbb{E}[X_i] = 0$, for all $i \in [1, \dots, p]$. Since the graph is a forest its distribution $p_G(\mathbf{x})$ belongs to $\mathcal{P}_T(0, \beta)$, for some interaction parameters θ . If \mathbf{Y} is the output of the BSC channel (in the hidden model) with distribution $p_{\dagger}(\mathbf{x})$, then $p_{\dagger}(\mathbf{x})$ belongs to $\mathcal{P}_T(0, \arctanh(c^2 \tanh \beta))$, the structure of the variables \mathbf{Y} is G and for the parameters θ' of $p_{\dagger}(\mathbf{x})$ we have $\theta'_{st} = \arctanh(c^2 \tanh(\theta_{st}))$, for all $(s, t) \in \mathcal{E}_G$.*

Proof We can start by giving the expression of any distribution of $\mathbf{x} \in \{-1, 1\}^p$: Using the indicator $1_{X_i=x_i} = \frac{1+x_i X_i}{2}$ we have

$$\begin{aligned} \mathbb{P}(\mathbf{X} = \mathbf{x}) &= \mathbb{E} \left[\prod_i 1_{X_i=x_i} \right] \\ &= 2^{-p} \left[1 + \sum_{i_1} x_{i_1} \mathbb{E}[X_{i_1}] + \sum_{i_1 < i_2} x_{i_1} x_{i_2} \mathbb{E}[X_{i_1} X_{i_2}] + \sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \mathbb{E}[X_{i_1} X_{i_2} X_{i_3}] \right. \\ &\quad \left. + \dots + x_1 x_2 \dots x_p \mathbb{E}[X_1 X_2 \dots X_p] \right]. \end{aligned} \quad (221)$$

We assume that a graph $G = (\mathcal{V}, \mathcal{E}_G)$, with $|\mathcal{V}| = p$, $|\mathcal{E}_G| = \frac{p}{2} \in \mathbb{N}$ has no edge connected to any other edge. We also assume that $\mathbb{E}[X_i] = 0, \forall i \in [1, \dots, p]$. Then (221) it could be

$$p_G(\mathbf{x}) = \frac{\prod_{i \in 2\mathbb{N}-1}^{p-1} (1 + x_i x_{i+1} \mathbb{E}[X_i X_{i+1}])}{2^p} \quad (222)$$

as a specific example, or in general it may be written as

$$p_G(\mathbf{x}) = \frac{\prod_{i \in 2\mathbb{N}-1}^{p-1} (1 + x_j x_{j+1} \mathbb{E}[X_j X_{j+1}])}{2^p} \quad (223)$$

where the set of integers $\{j_1, j_2, \dots, j_p\}$ can be any permutation of $\{1, 2, \dots, p\}$. Since its value of $i \in \{1, 3, 5, \dots, p-1\}$ corresponds to an edge we have $P_G(\mathbf{x}) \in \mathcal{P}_T(0, \beta)$. For the specific choice of G , by expanding (223) we find the equivalent expression of (221) as

$$\begin{aligned} p_G(\mathbf{x}) = \frac{1}{2^p} & \left[1 + \sum_{i \in 2\mathbb{N}-1}^{p-1} x_{j_i} x_{j_{i+1}} \mathbb{E}[X_s X_t] \right. \\ & + \sum_{i_1 < i_2; i_1, i_2 \in 2\mathbb{N}-1}^{p-1} x_{j_{i_1}} x_{j_{i_1+1}} x_{j_{i_2}} x_{j_{i_2+1}} \mathbb{E}[X_{j_{i_1}} X_{j_{i_1+1}}] \mathbb{E}[X_{j_{i_2}} X_{j_{i_2+1}}] \\ & \left. + \dots + \prod_{i \in 2\mathbb{N}-1}^{p-1} x_{j_i} x_{j_{i+1}} \mathbb{E} \left[\prod_{i=1: i \in 2\mathbb{N}-1}^{p-1} X_{j_i} X_{j_{i+1}} \right] \right], \end{aligned} \quad (224)$$

or equivalently by enumerating the edges we have

$$\begin{aligned} p_G(\mathbf{x}) = \frac{1}{2^p} & \left[1 + \sum_{(s,t) \in \mathcal{E}_G} x_s x_t \mathbb{E}[X_s X_t] + \sum_{(s,t)^2 \in \mathcal{P}(\mathcal{E}_G)} x_{s_{i_1}} x_{t_{i_1}} x_{s_{i_2}} x_{t_{i_2}} \mathbb{E}[X_{s_{i_1}} X_{t_{i_1}}] \mathbb{E}[X_{s_{i_2}} X_{t_{i_2}}] \right. \\ & \left. + \dots + \prod_{j=1}^{p/2} x_{s_{i_j}} x_{t_{i_j}} \mathbb{E} \left[\prod_{j=1}^{p/2} X_{s_{i_j}} X_{t_{i_j}} \right] \right], \end{aligned} \quad (225)$$

where $(s, t)^k = (s_{i_1}, t_{i_1}) \times (s_{i_2}, t_{i_2}) \times \dots \times (s_{i_k}, t_{i_k}) \in \mathcal{P}(\mathcal{E}_G)$. where the \mathcal{P} symbol is the power set.

For the noisy model that we consider (BSC) the distribution of the channel output variable $p_{\dagger}(\mathbf{y}) \notin \mathcal{P}_T(0, \beta)$ when $p(\mathbf{x}) \in \mathcal{P}_T(0, \beta)$ in the general case. However, when the structure is the G that is defined above we have that both distributions of the input and output belong to $\mathcal{P}_T(0, \beta)$. The probability mass function of \mathbf{Y} can be found as

$$p_{\dagger}(\mathbf{y}) = \sum_{\mathbf{x}, \mathbf{w}: x_i w_i = y_i} p_G(\mathbf{x}) p(\mathbf{w}), \quad (226)$$

where the probability mass function of the noise is denoted as $p(\mathbf{w})$. Then, we may write

$$\begin{aligned} & \sum_{\mathbf{x}, \mathbf{w}: x_i w_i = y_i} p_G(\mathbf{x}) p(\mathbf{w}) \\ = \frac{1}{2^p} & \sum_{\mathbf{x}, \mathbf{w}: x_i w_i = y_i} p(\mathbf{w}) \left[1 + \sum_{(s,t) \in \mathcal{E}_G} x_s x_t \mathbb{E}[X_s X_t] \right. \\ & + \sum_{(s,t)^2 \in \mathcal{P}(\mathcal{E}_G)} x_{s_{i_1}} x_{t_{i_1}} x_{s_{i_2}} x_{t_{i_2}} \mathbb{E}[X_{s_{i_1}} X_{t_{i_1}}] \mathbb{E}[X_{s_{i_2}} X_{t_{i_2}}] + \dots + \prod_{j=1}^{p/2} x_{s_{i_j}} x_{t_{i_j}} \mathbb{E}[X_{s_{i_j}} X_{t_{i_j}}] \left. \right] \\ = \frac{1}{2^p} & \left[1 + c^2 \sum_{(s,t) \in \mathcal{E}_G} y_s y_t \mathbb{E}[X_s X_t] + c^4 \sum_{(s,t)^2 \in \mathcal{P}(\mathcal{E}_G)} y_{s_{i_1}} y_{t_{i_1}} y_{s_{i_2}} y_{t_{i_2}} \mathbb{E}[X_{s_{i_1}} X_{t_{i_1}}] \mathbb{E}[X_{s_{i_2}} X_{t_{i_2}}] \right. \\ & \left. + \frac{1}{2^p} \left[+ \dots + c^p \prod_{j=1}^{p/2} y_{s_{i_j}} y_{t_{i_j}} \mathbb{E}[X_{s_{i_j}} X_{t_{i_j}}] \right] \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2^p} \left[1 + \sum_{(s,t) \in \mathcal{E}_G} y_s y_t \mathbb{E}[Y_s Y_t] + \sum_{(s,t)^2 \in \mathcal{P}(\mathcal{E}_G)} y_{s_{i_1}} y_{t_{i_1}} y_{s_{i_2}} y_{t_{i_2}} \mathbb{E}[Y_{s_{i_1}} Y_{t_{i_1}}] \mathbb{E}[Y_{s_{i_2}} Y_{t_{i_2}}] \right] \\
 &\quad + \frac{1}{2^p} \left[+ \dots + \prod_{j=1}^{p/2} y_{s_{i_j}} y_{t_{i_j}} \mathbb{E}[Y_{s_{i_j}} Y_{t_{i_j}}] \right] \\
 &= \frac{\prod_{i:\text{odd}}^{p-1} (1 + y_{j_i} y_{j_{i+1}} \mathbb{E}[Y_{j_i} Y_{j_{i+1}}])}{2^p}. \tag{227}
 \end{aligned}$$

Notice that the distribution of the output is identical to (223) when only the correlations change from μ_e to $\mu_e^\dagger = c^2 \mu_e$, for all $e \in G$ (and $c \triangleq \mathbb{E}[N_i] = 1 - 2q$, BSC). From the above result we conclude that the noisy variable \mathbf{Y} has the same structure G as \mathbf{X} with weights $\theta'_{st} = \arctanh(c^2 \tanh(\theta_{st}))$, for all $(s, t) \in \mathcal{E}$. \blacksquare

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