Spatially Controlled Relay Beamforming: 2-Stage Optimal Policies

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Abstract

The problem of enhancing Quality-of-Service (QoS) in power constrained, mobile relay beamforming networks, by optimally and dynamically controlling the motion of the relaying nodes, is considered, in a dynamic channel environment. We assume a time slotted system, where the relays update their positions before the beginning of each time slot. Modeling the wireless channel as a Gaussian spatiotemporal stochastic field, we propose a novel 2-stage stochastic programming problem formulation for optimally specifying the positions of the relays at each time slot, such that the expected QoS of the network is maximized, based on causal Channel State Information (CSI) and under a total relay transmit power budget. This results in a schema where, at each time slot, the relays, apart from optimally beamforming to the destination, also optimally, predictively decide their positions at the next time slot, based on causally accumulated experience. Exploiting either the Method of Statistical Differentials, or the multidimensional Gauss-Hermite Quadrature Rule, the stochastic program considered is shown to be approximately equivalent to a set of simple subproblems, which are solved in a distributed fashion, one at each relay. Optimality and performance of the proposed spatially controlled system are also effectively assessed, under a rigorous technical framework; strict optimality is rigorously demonstrated via the development of a version of the Fundamental Lemma of Stochastic Control, and, performance-wise, it is shown that, quite interestingly, the optimal average network QoS exhibits an increasing trend across time slots, despite our myopic problem formulation. Numerical simulations are presented, experimentally corroborating the success of the proposed approach and the validity of our theoretical predictions.

Keywords. Spatially Controlled Relay Beamforming, Mobile Relay Beamforming, Network Mobility Control, Network Utility Optimization, QoS Maximization, Motion Control, Distributed Cooperative Networks, Stochastic Programming.

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Contents

1 Introduction

Distributed, networked communication systems, such as relay beamforming networks [1–7] (e.g., Amplify & Forward (AF)) are typically designed without explicitly considering how the positions of the networking nodes might affect the quality of the communication. Optimum physical placement of assisting networking nodes, which could potentially improve the quality of the communication, does not constitute a clear network design aspect. However, in most practical settings in physical layer communications, the Channel State Information (CSI) observed by each networking node, per channel use, although (modeled as) random, it is both spatially and temporally correlated. It is, therefore, reasonable to ask if and how system performance could be improved by controlling the positions of certain network nodes, based on causal side information, and exploiting the spatiotemporal dependencies of the wireless medium.

Recently, autonomous node mobility has been proposed as an effective means to further enhance performance in various distributed network settings. In [8], optimal transmit AF beamforming has been combined with potential field based relay mobility control in multiuser cooperative networks, in order to minimize relay transmit power, while meeting certain Quality-of-Service (QoS) constraints. In [9], in the framework of information theoretic physical layer security, decentralized jammer motion control has been jointly combined with noise nulling and cooperative jamming, maximizing the network secrecy rate. In [10], optimal relay positioning has been studied in systems where multiple relays deliver information to a destination, in the presence of an eavesdropper, with a goal of maximizing or achieving a target level of ergodic secrecy.

In the complementary context of communication aware (comm-aware) robotics, node mobility has been exploited in distributed robotic networks, in order to enhance system performance, in terms of maintaining reliable, in-network communication connectivity [11–14], and optimizing network energy management [15]. Networked node motion control has also been exploited in special purpose applications, such as networked robotic surveillance [16] and target tracking [17].

In [8–10], the links among the nodes of the network (or the related statistics) are assumed to be available in the form of static channel maps, during the whole motion of the jammers/relays. However, this is an oversimplifying assumption in scenarios where the channels change significantly in time and space [18–20].

In this paper, we try to overcome this major limitation, and we consider the problem of optimally and dynamically updating relay positions in one source/destination relay beamforming networks, *in a dynamic channel environment*. Different from [8–10], we model the wireless channel as a *spatiotemporal stochastic field*; this approach may be seen as a versatile extension of a realistic, commonly employed "log-normal" channel model [20]. We then propose a 2-*stage stochastic programming* problem formulation, optimally specifying the positions of the relays *at each time slot*, such that the *Signal-to-Interference+Noise Ratio (SINR)* or *QoS* at the destination, at the same time slot, is maximized *on average*, based on *causal CSI*, and subject to a *total power constraint at the relays*. At each time slot, the relays not only beamform to the destination, but also optimally, *predictively* decide their positions at the next time slot, based on their experience (causal actions and channel observations). This novel, cyber-physical system approach to relay beamforming is termed as *Spatially Controlled Relay Beamforming*.

Exploiting the assumed stochastic channel structure, it is first shown that the proposed optimal motion control problem is equivalent to a set of simpler, two dimensional subproblems, which can be solved in a *distributed fashion*, one at each relay, *without the need for intermediate exchange of messages* among the relays. However, each of the objectives of the aforementioned subproblems involves the evaluation of a conditional expectation of a well defined ratio of almost surely positive random variables, which is *impossible to perform analytically*, calling for the development of easily implementable approximations to each of the original problems. Two such heuristics are considered. The first is based on the so-called *Method of Statistical Differentials* [21], whereas the second constitutes a brute force approach, based on the *multidimensional Gauss-Hermite Quadrature Rule*, a readily available routine for numerical integration. In both cases, the original problem objective is replaced by the respective approximation, which, in both cases, is shown to be easily computed via simple, closed form expressions. The computational complexity of both approaches is also discussed and characterized. Subsequently, we present an important result, along with the respective detailed technical development, characterizing the performance of the proposed system, *across time slots* (Theorems 6 and 7). In a nutshell, this result states that, although our *problem objective is itself myopic* at each time slot, the *expected network QoS exhibits an increasing trend across time slots* (in other words, the expected QoS increases in time, within a small positive slack), under optimal decision making at the relays. Lastly, we present representative numerical simulations, experimentally confirming both the efficacy and feasibility of the proposed approach, as well as the validity of our theoretical predictions.

During exposition of the proposed spatially controlled relay beamforming system, we concurrently develop and utilize a rigorous discussion concerning the optimality of our approach, and with interesting results (Section 8.2 / Appendix B). Clearly, our problem formulation is challenging; it involves a *variational* stochastic optimization problem, where, at each time slot, the decision variable, *a function* of the so far available useful information in the system (also called a *policy*, or a *decision rule*), *constitutes itself the spatial coordinates, from which every network relay will observe the underlying spatiotemporal channel field, at the next time slot.* In other words, our formulation requires solving an (*myopic*, in particular) *optimal spatial field sampling problem, in a dynamic fashion.* Such a problem raises certain fundamental questions, not only related to our proposed spatially controlled beamforming formulation, but also to a large class of variational stochastic programs of similar structure.

In this respect, our contributions are partially driven by assuming an underlying complete base probability space of otherwise *arbitrary structure*, generating all random phenomena considered in this work. Under this general setting, we explicitly identify sufficient conditions, which guarantee the validity of the so-called *substitution rule for conditional expectations*, specialized to such expectations of random spatial (in general) fields/functions *with an also random spatial parameter*, relative to some σ-algebra, which makes the latter parameter measurable (fixed) (Definition 6 & Theorem 8). General validity of the substitution rule, without imposing additional, special conditions, traces back to the existence of regular conditional distributions, defined *directly* on the sample space of the underlying base probability space. Such regular conditional distributions cannot be guaranteed to exist, unless the sample space has nice topological properties, for instance, if it is Polish [22]. In the context of our spatially controlled beamforming application, such structural requirements on the sample space, which, by assumption, is conceived as a model of "nature", and generates the spatiotemporal channel field sampled by the relays, are simply not reasonable. Considering this, our first contribution is to show that it is possible to guarantee the validity of the form of the substitution rule under consideration by imposing conditions on the topological structure of the involved random field, rather than that of the sample space (a part of its domain). This results in a rather generally applicable problem setting (Theorem 8).

In this work, the validity of the substitution rule is ascertained by imposing simple continuity assumptions on the random functions involved, which, in some cases, might be considered somewhat restrictive. Nevertheless, those assumptions can be significantly weakened, guaranteeing the validity of the substitution rule for vastly discontinuous random functions, including, for instance, cases with random discontinuities, or random jumps. The development of this extended analysis, though, is out of the scope of this paper, and will be presented elsewhere.

The validity of the substitution rule is vitally important in the treatment of a wide class of variational stochastic programs, including that involved in the proposed spatially controlled beamforming approach. In particular, leveraging the power of the substitution rule, we develop a version of the so-called *Fundamental Lemma of Stochastic Control (FLSC)* [23–28] (Lemma 3), which provides sufficient conditions that permit interchange of integration (expectation) and max/minimization in general variational (stochastic) programming settings. The FLSC allows the initial variational problem to be *exchanged* by a related, though *pointwise* (ordinary) optimization problem, thus efficiently reducing the search over a class of functions (initial problem) to searching over constants, which is, of course, a standard and much more handleable optimization setting. In slightly different ways, the FLSC is evidently utilized in relevant optimality analysis both in Stochastic Programming [25, 26], and in Dynamic Programming & Stochastic Optimal Control [23, 24, 27, 28].

A very general version of the FLSC is given in ([25], Theorem 14.60), where unconstrained variational optimization of integrals of extended real-valued *random lower semicontinuous functions* [26], or, by another name, *normal integrands* [25], with respect to a general σ-finite measure, is considered. Our version of the FLSC may be considered a useful variation of Theorem 14.60 in [25], and considers *constrained* variational optimization problems involving integrals of random functions, but with respect to some base *probability* measure (that is, expectations). In our result, via the tower property of expectations, the role of the normal integrand in ([25], Theorem 14.60) is played by the conditional expectation of the random function considered, relative to a σ-algebra, which makes the respective decision variable of the problem (a function(al)) measurable. Assuming a base probability space of arbitrary structure, this argument is justified by assuming validity of the substitution rule, which, in turn, is ascertained under our previously developed sufficient conditions. Different from ([25], Theorem 14.60), in our version of the FLSC, apart from natural Borel measurability requirements, *no continuity assumptions are directly imposed* on the structure on either the random function, or the respective conditional expectation. In this respect, our result extends ([25], Theorem 14.60), and is of independent interest.

On the other hand, from the strongly related perspective of Stochastic Optimal Control, our version of the FLSC may be considered as the basic building block for further development of Bellman Equation-type, Dynamic Programming solutions [27, 28], under a strictly Borel measurability framework, sufficient for our purposes. Quite differently though, in our formulation, the respective cost (at each stage of the problem) is itself a random function (a spatial field), whose domain is the Cartesian product of a base space of arbitrary topology, with another, nicely behaved Borel space, instead of the usual Cartesian product of two Borel spaces (the spaces of state and controls), as in the standard dynamic programming setting [27, 28]. Essentially, our formulation is "one step back" as compared to the basic dynamic programming model of [27, 28], in the sense that the cost considered herein refers *directly* back to the base space. As a result, different treatment of the problem is required; essentially, the validity of the substitution rule for our cost function bypasses the requirement for existence of conditional distributions, and exploits potential nice properties of the respective conditional cost (in our case, joint *Borel* measurability).

Emphasizing on our particular problem formulation, our functional assumptions, which guarantee the validity of the substitution rule, combined with the FLSC, result in a total of six sufficient conditions, under which strict optimality via problem exchangeability is guaranteed (conditions **C1-C6** in Lemma 4). Those conditions are subsequently shown to be satisfied specifically for the spatially controlled beamforming problem under consideration (verification Theorem 3), ensuring strict optimality of a solution obtained by exploiting problem exchangeability.

Finally, motivated by the need to provide performance guarantees for the proposed myopic stochastic decision making scheme (our spatially controlled beamforming network), we introduce the concept of a *linear martingale difference generator* spatiotemporal field. We then rigorously show that, when such fields are involved in the objective of a *myopic* stochastic sampling scheme, *stagewise myopic stochastic exploration of the involved field is, under conditions, either monotonic, or quasi-monotonic* (that is, monotonic within some small positive slack)*, either under optimal sampling, or when retaining the same sampling policy next.* This result is the basis for providing performance guarantees for the proposed spatially controlled relay beamforming system, as briefly stated above.

Notation (some and basic): Matrices and vectors will be denoted by boldface uppercase and boldface lowercase letters, respectively. Calligraphic letters and formal script letters will denote sets and σ-algebras, respectively. The operators $(\cdot)^{\boldsymbol{T}}$ and $(\cdot)^{\boldsymbol{H}}$, $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ will denote transposition, conjugate transposition, minimum and maximum eigenvalue, respectively. The ℓ_p-norm of $\boldsymbol{x} \in \mathbb{R}^n$ is $\|\boldsymbol{x}\|_p \triangleq \left(\sum_{i=1}^n |x(i)|^p\right)^{1/p}$, for all $\mathbb{N} \ni p \geq 1$. For any $\mathbb{N} \ni N \geq 1$, $\mathbb{S}^N$, $\mathbb{S}_+^N$, $\mathbb{S}_{++}^N$ will denote the sets of symmetric, symmetric positive semidefinite and symmetric positive definite matrices, respectively. The finite N-dimensional identity operator will be denoted as $\mathbf{I}_N$. Additionally, we define $\mathfrak{J} \triangleq \sqrt{-1}$, $\mathbb{N}^+ \triangleq \{1,2,\ldots\}$, $\mathbb{N}_n^+ \triangleq \{1,2,\ldots,n\}$, $\mathbb{N}_n \triangleq \{0\} \cup \mathbb{N}_n^+$ and $\mathbb{N}_n^m \triangleq \mathbb{N}_n^+ \setminus \mathbb{N}_{m-1}^+$, for positive naturals $n > m$.

2 System Model

On a compact, square planar region $\mathcal{W} \subset \mathbb{R}^2$, we consider a wireless cooperative network consisting of one source, one destination and $R \in \mathbb{N}^+$ assistive relays, as shown in Fig. 2.1. Each entity of the network is equipped with a single antenna, being able for both information reception and broadcasting/transmission. The source and destination are stationary and located at $\mathbf{p}_S \in \mathcal{W}$ and $\mathbf{p}_D \in \mathcal{W}$, respectively, whereas the relays are assumed to be mobile; each relay $i \in \mathbb{N}_R^+$ moves along a trajectory $\mathbf{p}_i(t) \in \mathcal{S} \subset \mathcal{W} - \{\mathbf{p}_S, \mathbf{p}_D\} \subset \mathcal{W}$, where, in general, $t \in \mathbb{R}_+$, and where $\mathcal{S}$ is compact. We also define the supervector $\mathbf{p}(t) \triangleq \left[\mathbf{p}_1^{\boldsymbol{T}}(t)\ \mathbf{p}_2^{\boldsymbol{T}}(t)\ \ldots\ \mathbf{p}_R^{\boldsymbol{T}}(t)\right]^{\boldsymbol{T}} \in \mathcal{S}^R \subset \mathbb{R}^{2R\times 1}$. Additionally, we assume that the relays can cooperate with each other, either by exchanging local messages, or by communicating with a local fusion center, through a dedicated channel. Hereafter, as already stated above, all probabilistic arguments made below presume the existence of a complete base probability space of otherwise completely arbitrary structure, prespecified by a triplet $(\Omega, \mathscr{F}, \mathcal{P})$. This base space models a universal source of randomness, generating all stochastic phenomena in our considerations.

Assuming that a direct link between the source and the destination does not exist, the role of the relays is determined to be assistive to the communication, operating in a classical, two phase AF relaying mode. Fix a $T > 0$, and *divide the time interval* $[0, T]$ *into* N_T *time slots*, with $t \in \mathbb{N}_{N_T}^+$ denoting the respective time slot. Let $s(t) \in \mathbb{C}$, with $\mathbb{E}\left\{|s(t)|^2\right\} \equiv 1$, denote the symbol to be transmitted at time slot t. Also, assuming a flat fading channel model, as well as channel reciprocity and quasistaticity in each time slot, let the sets $\{f_i(t) \in \mathbb{C}\}_{i \in \mathbb{N}_R^+}$ and $\{g_i(t) \in \mathbb{C}\}_{i \in \mathbb{N}_R^+}$ contain the *random, spatiotemporally varying* source-relay and relay-destination channel gains, respectively. These are further assumed to be *evaluations* of the *separable random channel fields* or *maps* $f(\mathbf{p}, t)$ and $g(\mathbf{p}, t)$, respectively, that is, $f_i(t) \equiv f(\mathbf{p}_i(t), t)$ and $g_i(t) \equiv g(\mathbf{p}_i(t), t)$, for all $i \in \mathbb{N}_R^+$ and for all $t \in \mathbb{N}_{N_T}^+$. Then, if $P_0 > 0$ denotes the transmission power of the source, during AF phase 1, the signals received at the relays can be expressed as

$$r_i(t) \triangleq \sqrt{P_0} f_i(t)\, s(t) + n_i(t) \in \mathbb{C}, \tag{2.1}$$

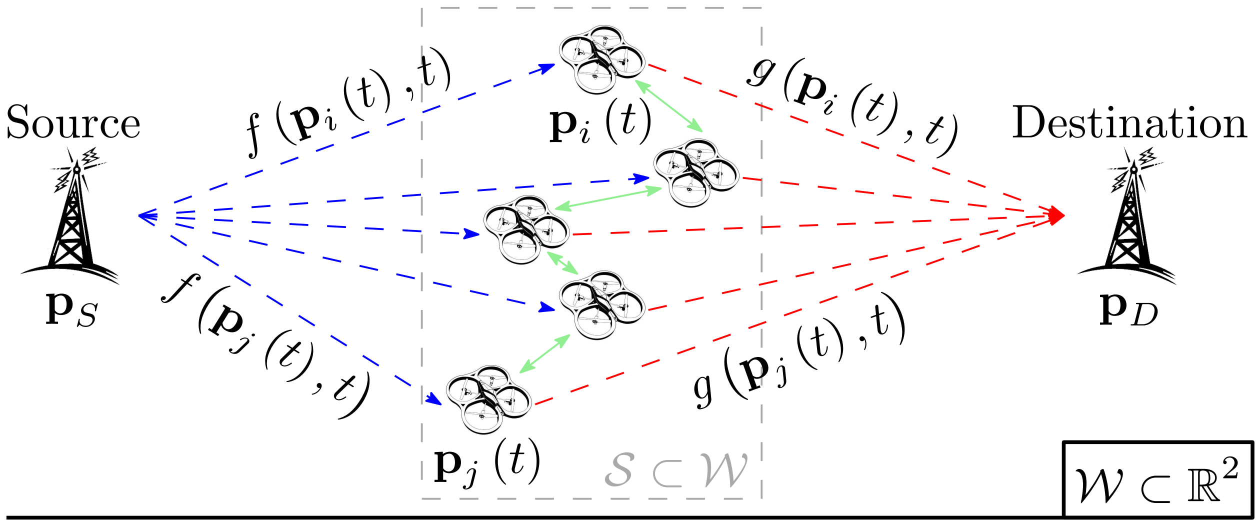

Figure 2.1: A schematic of the system model considered.

for all $i \in \mathbb{N}_R^+$ and for all $t \in \mathbb{N}_{N_T}^+$, where $n_i(t) \in \mathbb{C}$, with $\mathbb{E}\left\{|n_i(t)|^2\right\} \equiv \sigma^2$, constitutes a zero mean observation noise process at the i-th relay, independent across relays. During AF phase 2, all relays simultaneously retransmit the information received, each modulating their received signal by a weight $w_i(t) \in \mathbb{C}, i \in \mathbb{N}_R^+$. The signal received at the destination can be expressed as

$$\begin{aligned} y(t) &\triangleq \sqrt{P_0} \sum_{i \in \mathbb{N}_R^+} w_i(t)\, g_i(t)\, r_i(t) \\ &\equiv \underbrace{\sqrt{P_0} \sum_{i \in \mathbb{N}_R^+} w_i(t)\, g_i(t)\, f_i(t)\, s(t)}_{\text{signal (transformed)}} + \underbrace{\sum_{i \in \mathbb{N}_R^+} w_i(t)\, g_i(t)\, n_i(t) + n_D(t)}_{\text{interference + reception noise}} \in \mathbb{C}, \end{aligned} \tag{2.2}$$

for all $i \in \mathbb{N}_R^+$ and $t \in \mathbb{N}_{N_T}^+$, where $n_D(t) \in \mathbb{C}$, with $\mathbb{E}\left\{|n_D(t)|^2\right\} \equiv \sigma_D^2$, constitutes a zero mean, spatiotemporally white noise process at the destination.

In the following, it is assumed that the channel fields $f(\mathbf{p}, t)$ and $g(\mathbf{p}, t)$ may be *statistically dependent both spatially and temporally*, and that, *as usual*, the processes $s(t)$, $[f(\mathbf{p}, t)\ g(\mathbf{p}, t)]$, $n_i(t)$ for all $i \in \mathbb{N}_R^+$, and $n_D(t)$ are mutually independent. Also, we will assume that, at each time slot t, CSI $\{f_i(t)\}_{i \in \mathbb{N}_R^+}$ and $\{g_i(t)\}_{i \in \mathbb{N}_R^+}$ is known *exactly* to all relays. This may be achieved through pilot based estimation.

3 Spatiotemporal Wireless Channel Modeling

This section introduces a general stochastic model for describing the spatiotemporal evolution of the wireless channel. For the benefit of the reader, a more intuitive justification of this general model is also provided. Additionally, some extensions to the model are briefly discussed, highlighting its versatility, along with some technical considerations, which will be of importance later, for analyzing the theoretical consistency of the subsequently proposed techniques.

3.1 Large Scale Gaussian Channel Modeling in the dB Domain

At each space-time point $(\mathbf{p}, t) \in \mathcal{S} \times \mathbb{N}^+_{N_T}$, the source-relay channel field may be decomposed as the product of three space-time varying components [29], as

$$f(\mathbf{p}, t) \equiv \underbrace{f^{PL}(\mathbf{p})}_{\text{path loss}} \underbrace{f^{SH}(\mathbf{p}, t)}_{\text{shadowing}} \underbrace{f^{MF}(\mathbf{p}, t)}_{\text{fading}} e^{\mathfrak{J} \frac{2\pi \|\mathbf{p} - \mathbf{p}_S\|_2}{\lambda}}, \tag{3.1}$$

where $\mathfrak{J} \triangleq \sqrt{-1}$ denotes the imaginary unit, $\lambda > 0$ denotes the wavelength employed for the communication, and:

1. $f^{PL}(\mathbf{p}) \triangleq \|\mathbf{p} - \mathbf{p}_S\|_2^{-\ell/2}$ is the *path loss field*, a deterministic quantity, with $\ell > 0$ being the path loss exponent.
2. $f^{SH}(\mathbf{p}, t) \in \mathbb{R}$ is the *shadowing field*, whose square is, for each $(\mathbf{p}, t) \in \mathcal{S} \times \mathbb{N}^+_{N_T}$, a base-10 log-normal random variable with zero location.
3. $f^{MF}(\mathbf{p}, t) \in \mathbb{C}$ constitutes the *multipath fading field*, a stationary process with known statistics.

The same decomposition holds in direct correspondence for the relay-destination channel field, $g(\mathbf{p}, t)$. Additionally, if "$\perp\!\!\!\perp$" means "is statistically independent of", it is assumed that [20]

$$\left[f^{MF}(\mathbf{p}, t)\, g^{MF}(\mathbf{p}, t)\right] \perp\!\!\!\perp \left[f^{SH}(\mathbf{p}, t)\, g^{SH}(\mathbf{p}, t)\right] \quad \text{and} \tag{3.2}$$

$$f^{MF}(\mathbf{p}, t) \perp\!\!\!\perp g^{MF}(\mathbf{p}, t). \tag{3.3}$$

In particular, if the phase of $f^{MF}(\mathbf{p}, t)$ is denoted as $\phi_f(\mathbf{p}, t) \in [-\pi, \pi]$, is further assumed that

$$\left|f^{MF}(\mathbf{p}, t)\right| \perp\!\!\!\perp \phi_f(\mathbf{p}, t), \tag{3.4}$$

and the same for for $g^{MF}(\mathbf{p}, t)$. It also follows that

$$\left[\left|f^{MF}(\mathbf{p}, t)\right| \left|g^{MF}(\mathbf{p}, t)\right|\right] \perp\!\!\!\perp \left[f^{SH}(\mathbf{p}, t)\, g^{SH}(\mathbf{p}, t)\right]. \tag{3.5}$$

We are interested in the magnitudes of both fields $f(\mathbf{p}, t)$ and $g(\mathbf{p}, t)$. Instead of working with the multiplicative model described by (3.1), it is much preferable to work in logarithmic scale. We may define the *log-scale magnitude field*

$$F(\mathbf{p}, t) \triangleq \alpha_S(\mathbf{p})\, \ell + \sigma_S(\mathbf{p}, t) + \xi_S(\mathbf{p}, t), \tag{3.6}$$

where we define

$$-\alpha_S(\mathbf{p}) \triangleq 10 \log_{10}(\|\mathbf{p} - \mathbf{p}_S\|_2), \tag{3.7}$$

$$\sigma_S(\mathbf{p}, t) \triangleq 10 \log_{10}\left(f^{SH}(\mathbf{p}, t)\right)^2 \quad \text{and} \tag{3.8}$$

$$\xi_S(\mathbf{p}, t) \triangleq 10 \log_{10}\left|f^{MF}(\mathbf{p}, t)\right|^2 - \rho, \quad \text{with} \tag{3.9}$$

$$\rho \triangleq \mathbb{E}\left\{10 \log_{10}\left|f^{MF}(\mathbf{p}, t)\right|^2\right\}, \tag{3.10}$$

for all $(\mathbf{p},t)\in\mathcal{S}\times\mathbb{N}^{+}_{N_T}$. It is then trivial to show that the magnitude of $f(\mathbf{p},t)$ may be reconstructed via the *bijective* formula

$$|f(\mathbf{p},t)|\equiv 10^{\rho/20}\exp\left(\frac{\log(10)}{20}F(\mathbf{p},t)\right), \tag{3.11}$$

for all $(\mathbf{p},t)\in\mathcal{S}\times\mathbb{N}^{+}_{N_T}$, a "trick" that will prove very useful in the next section. Regarding $g(\mathbf{p},t)$, the log-scale field $G(\mathbf{p},t)$ is defined in the same fashion, but replacing the subscript "S" by "D".

For each relay $i\in\mathbb{N}^{+}_{R}$, let us define the respective log-scale channel magnitude processes $F_i(t)\triangleq F(\mathbf{p}_i(t),t)$ and $G_i(t)\triangleq G(\mathbf{p}_i(t),t)$, for all $t\in\mathbb{N}^{+}_{N_T}$. Of course, we may stack all the $F_i(t)$'s defined in (3.6), resulting in the vector additive model

$$\boldsymbol{F}(t)\triangleq\boldsymbol{\alpha}_S(\mathbf{p}(t))\,\ell+\boldsymbol{\sigma}_S(t)+\boldsymbol{\xi}_S(t)\in\mathbb{R}^{R\times 1}, \tag{3.12}$$

where $\boldsymbol{\alpha}_S(t)$, $\boldsymbol{\sigma}_S(t)$ and $\boldsymbol{\xi}_S(t)$ are defined accordingly. We can also define $\boldsymbol{G}(t)\triangleq\boldsymbol{\alpha}_D(\mathbf{p}(t))\,\ell+\boldsymbol{\sigma}_D(t)+\boldsymbol{\xi}_D(t)\in\mathbb{R}^{R\times 1}$, with each quantity in direct correspondence with (3.12). We may also define, in the same manner, the log-scale shadowing and multipath fading processes $\sigma^{i}_{S(D)}(t)\triangleq\sigma_{S(D)}(\mathbf{p}_i(t),t)$ and $\xi^{i}_{S(D)}(t)\triangleq\xi_{S(D)}(\mathbf{p}_i(t),t)$, for all $t\in\mathbb{N}^{+}_{N_T}$, respectively.

Next, let us focus on the spatiotemporal dynamics of $\{|f_i(t)|\}_i$ and $\{|g_i(t)|\}_i$, which are modeled through those of the shadowing components of $\{F_i(t)\}_i$ and $\{G_i(t)\}_i$. It is assumed that, for any N_T and any *deterministic* ensemble of positions of the relays in $\mathbb{N}^{+}_{N_T}$, say $\{\mathbf{p}(t)\}_{t\in\mathbb{N}^{+}_{N_T}}$, the random vector

$$\left[\boldsymbol{F}^{\boldsymbol{T}}(1)\;\boldsymbol{G}^{\boldsymbol{T}}(1)\;\ldots\;\boldsymbol{F}^{\boldsymbol{T}}(N_T)\;\boldsymbol{G}^{\boldsymbol{T}}(N_T)\right]^{\boldsymbol{T}}\in\mathbb{R}^{2RN_T\times 1} \tag{3.13}$$

is *jointly Gaussian* with known means and known covariance matrix. More specifically, on a per node basis, we let $\xi^{i}_{S(D)}(t)\overset{i.i.d.}{\sim}\mathcal{N}\left(0,\sigma^2_\xi\right)$ and $\sigma^{i}_{S(D)}(t)\overset{i.d.}{\sim}\mathcal{N}\left(0,\eta^2\right)$, for all $t\in\mathbb{N}^{+}_{N_T}$ and $i\in\mathbb{N}^{+}_{R}$ [20, 30]. In particular, extending Gudmundson's model [31] in a straightforward way, we propose defining the spatiotemporal correlations of the shadowing part of the channel as

$$\mathbb{E}\left\{\sigma^{i}_{S}(k)\,\sigma^{j}_{S}(l)\right\}\triangleq\eta^2\exp\left(-\frac{\left\|\mathbf{p}_i(k)-\mathbf{p}_j(l)\right\|_2}{\beta}-\frac{|k-l|}{\gamma}\right), \tag{3.14}$$

and correspondingly for $\left\{\sigma^{i}_{D}(t)\right\}_{i\in\mathbb{N}^{+}_{R}}$, and additionally,

$$\mathbb{E}\left\{\sigma^{i}_{S}(k)\,\sigma^{j}_{D}(l)\right\}\triangleq\mathbb{E}\left\{\sigma^{i}_{S}(k)\,\sigma^{j}_{S}(l)\right\}\exp\left(-\frac{\|\mathbf{p}_S-\mathbf{p}_D\|_2}{\delta}\right), \tag{3.15}$$

for all $(i,j)\in\mathbb{N}^{+}_{R}\times\mathbb{N}^{+}_{R}$ and for all $(k,l)\in\mathbb{N}^{+}_{N_T}\times\mathbb{N}^{+}_{N_T}$. In the above, $\eta^2>0$ and $\beta>0$ are called the *shadowing power* and the *correlation distance*, respectively [31]. In this fashion, we will call $\gamma>0$ and $\delta>0$ the *correlation time* and the *BS (Base Station) correlation*, respectively. For later reference, let us define the (cross)covariance matrices

$$\boldsymbol{\Sigma}_{SD}(k,l)\triangleq\mathbb{E}\left\{\boldsymbol{\sigma}_S(k)\,\boldsymbol{\sigma}^{\boldsymbol{T}}_D(l)\right\}+\mathbb{1}_{\{S\equiv D\}}\mathbb{1}_{\{k\equiv l\}}\sigma^2_\xi\mathbf{I}_R\in\mathbb{S}^{R}, \tag{3.16}$$

as well as

$$\boldsymbol{\Sigma}(k,l)\triangleq\begin{bmatrix}\boldsymbol{\Sigma}_{SS}(k,l) & \boldsymbol{\Sigma}_{SD}(k,l)\\ \boldsymbol{\Sigma}_{SD}(k,l) & \boldsymbol{\Sigma}_{DD}(k,l)\end{bmatrix}\in\mathbb{S}^{2R}, \tag{3.17}$$

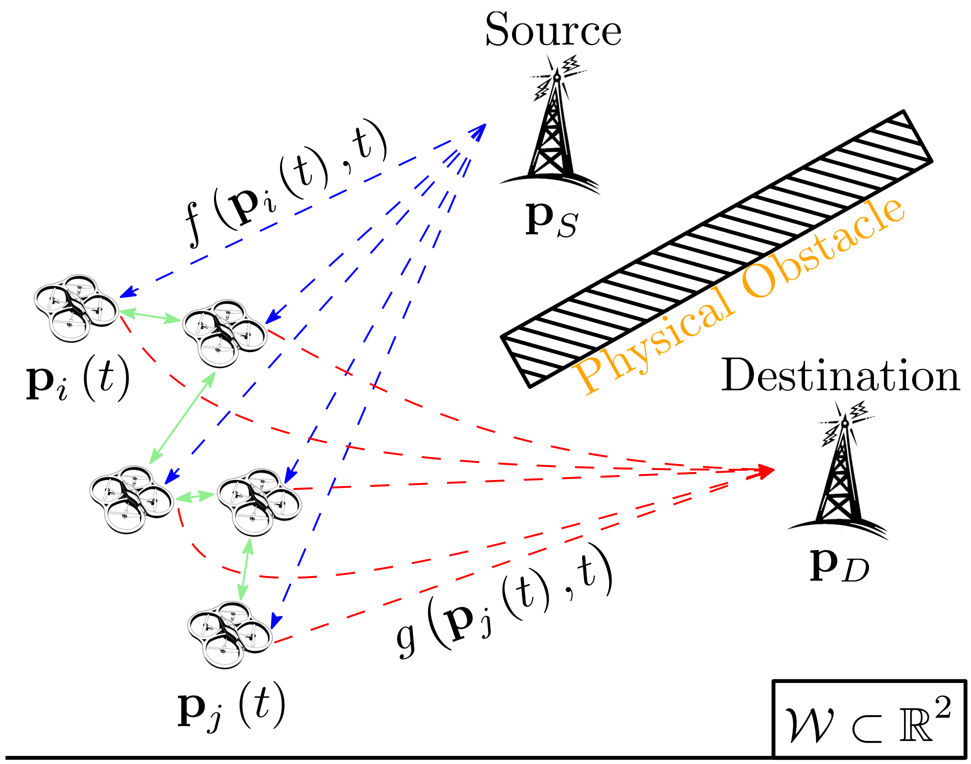

Figure 3.1: A case where source-relay and relay-destination links are likely to be correlated.

for all $(k,l) \in \mathbb{N}_{N_T}^+ \times \mathbb{N}_{N_T}^+$. Using these definitions, the covariance matrix of the joint distribution describing (3.13) can be readily expressed as

$$\mathbf{\Sigma} \triangleq \begin{bmatrix} \mathbf{\Sigma}(1,1) & \mathbf{\Sigma}(1,2) & \dots & \mathbf{\Sigma}(1,N_T) \\ \mathbf{\Sigma}(2,1) & \mathbf{\Sigma}(2,2) & \dots & \mathbf{\Sigma}(2,N_T) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Sigma}(N_T,1) & \mathbf{\Sigma}(N_T,2) & \cdots & \mathbf{\Sigma}(N_T,N_T) \end{bmatrix} \in \mathbb{S}^{2RN_T}. \tag{3.18}$$

Of course, in order for $\mathbf{\Sigma}$ to be a valid covariance matrix, it must be at least positive semidefinite, that is, in $\mathbb{S}_+^{2RN_T}$. If fact, for nearly all cases of interest, $\mathbf{\Sigma}$ is guaranteed to be strictly positive definite (or in $\mathbb{S}_{++}^{2RN_T}$), as the following result suggests.

Lemma 1. (Positive (Semi)Definiteness of $\mathbf{\Sigma}$) *For all possible deterministic trajectories of the relays on $\mathcal{S}^R \times \mathbb{N}_{N_T}^+$, it is true that $\mathbf{\Sigma} \in \mathbb{S}_{++}^{2RN_T}$, as long as $\sigma_\xi^2 \neq 0$. Otherwise, $\mathbf{\Sigma} \in \mathbb{S}_+^{2RN_T}$. In other words, as long as multipath (small-scale) fading is present in the channel response, the joint Gaussian distribution of the channel vector in (3.13) is guaranteed to be nonsingular.*

Proof of Lemma 1. See Appendix A. ∎

3.2 Model Justification

As already mentioned, the spatial dependence among the source-relay and relay-destination channel magnitudes (due to shadowing) is described via Gudmundson's model [31] (position related component in (3.14)), which has been very popular in the literature and also experimentally verified

[20, 31, 32]. Second, the Laplacian type of temporal dependence among the same groups of channel magnitudes also constitutes a reasonable choice, in the sense that channel magnitudes are expected to be significantly correlated only for small time lags, whereas, for larger time lags, such dependence should decay at a fast rate. For an experimental justification of the adopted model, see, for instance, [33, 34]. Also note that, this exponential temporal correlation model may result as a reformulation of Gudmundson's model, as well. Of course, one could use any other positive (semi)definite kernel, multiplying the spatial correlation exponential kernel, without changing the statement and proof of Lemma 1. Third, the incorporation of the spherical/isotropic BS correlation term in our proposed general model (in (3.15)) can be justified by the the existence of important cases where the source and destination might be close to each other and yet no direct link may exist between them. See, for instance, Fig. (3.1), where a "large" physical obstacle makes the direct communication between the source and the destination impossible. Then, relay beamforming can be exploited in order to enable efficient communication between the source and the destination, making intelligent use of the available resources, in order to improve or maintain a certain QoS in the network. In such cases, however, it is very likely that the shadowing parts of the source-relay and relay-destination links will be spatially and/or temporally correlated among each other, since shadowing is very much affected by the spatial characteristics of the terrain, which, in such cases, is common for both beamforming phases. Of course, by taking the BS station correlation $\delta \to 0$, one recovers the generic/trivial case where the source-relay and relay-destination links are *mutually independent*.

3.3 Extensions & Some Technical Considerations

It should be also mentioned that our general description of the wireless channel as a spatiotemporal Gaussian field, does not limit the covariance matrix $\mathbf{\Sigma}$ to be formed as in (3.18); other choices for $\mathbf{\Sigma}$ will work fine in our subsequent developments, as long as, *for each fixed* $t \in \mathbb{N}_{N_T}^+$, some mild conditions on the *spatial interactions* of the fields $\sigma_{S(D)}(\mathbf{p}, t)$ and $\xi_{S(D)}(\mathbf{p}, t)$, are satisfied. In what follows, we consider only the source-relay fields $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$. The same arguments hold for the relay-destination fields $\sigma_D(\mathbf{p}, t)$ and $\xi_D(\mathbf{p}, t)$, in direct correspondence.

Fix $t \in \mathbb{N}_{N_T}^+$. Recall that, so far, we have defined the statistical behavior of both $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$ *only on a per-node basis*. However, since the spatiotemporal statistical model introduced in Section 3.1 is assumed to be valid for any possible trajectory of the relays in $\mathcal{S}^R \times \mathbb{N}_{N_T}^+$, each relay is allowed to be anywhere in $\mathcal{S}$, at each time slot t. This statistical construction *induces* the statistical structure (the *laws*) of both fields $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$ on $\mathcal{S}$.

As far as $\sigma_S(\mathbf{p}, t)$ is concerned, it is straightforward to see that it constitutes a Gaussian process with zero mean, and a continuous and *isotropic* covariance kernel $\mathbf{\Sigma}_\sigma : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$\mathbf{\Sigma}_\sigma(\boldsymbol{\tau}) \triangleq \eta^2 \exp\left(-\frac{\|\boldsymbol{\tau}\|_2}{\beta}\right), \tag{3.19}$$

where $\boldsymbol{\tau} \triangleq \mathbf{p} - \mathbf{q} \geq 0$, for all $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}^2$, which agrees with the model introduced in (3.14), for $k \equiv l$ (Gudmundson's model). Thus, $\sigma_S(\mathbf{p}, t)$ is a well defined random field.

However, this is not the case with $\xi_S(\mathbf{p}, t)$. Under no additional restrictions, $\xi_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{q}, t)$ are implicitly assumed to be independent for all $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}^2$, such that $\mathbf{p} \neq \mathbf{q}$. Thus, we are led to consider $\xi_S(\mathbf{p}, t)$ as a zero-mean white process in continuous space. However, it is well known that such a process is technically problematic in a measure theoretic framework. Nevertheless, we may observe that it is *not* actually essential to characterize the covariance structure of $\xi_S(\mathbf{p}, t)$ *for all* $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}^2$, with $\mathbf{p} \neq \mathbf{q}$. This is due to the fact that, at each time slot $t \in \mathbb{N}_{N_T}^+$, it is *physically* *impossible* for any two relays to be arbitrarily close to each other. We may thus make the following simple assumption on the positions of the relays, at each time slot $t \in \mathbb{N}^+_{N_T}$.

Assumption 1. (Relay Separation) *There exists an $\varepsilon_{MF} > 0$, such that, for all $t \in \mathbb{N}^+_{N_T}$ and any ensemble of relay positions at time slot t, $\{\mathbf{p}_i(t)\}_{i\in\mathbb{N}^+_R}$, it is true that*

$$\inf_{\substack{(i,j)\in\mathbb{N}^+_R\times\mathbb{N}^+_R \\ \textit{with } i\neq j}} \left\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\right\|_2 > \varepsilon_{MF}. \tag{3.20}$$

Assumption 1 simply states that, at each $t \in \mathbb{N}^+_{N_T}$, all relays are at least ε_{MF} distance units apart from each other. If this constraint is satisfied, then, without any loss of generality, we may define $\xi_S(\mathbf{p}, t)$ as a Gaussian field with zero mean, and with *any* continuous, isotropic (say) covariance kernel $\mathbf{\Sigma}_\xi : \mathbb{R}^2 \to \mathbb{R}$, which satisfies

$$\mathbf{\Sigma}_\xi(\boldsymbol{\tau}) \triangleq \begin{cases} \sigma^2_\xi, & \text{if } \boldsymbol{\tau} \equiv \mathbf{0} \\ 0, & \text{if } \|\boldsymbol{\tau}\|_2 \geq \varepsilon_{MF} \end{cases}, \tag{3.21}$$

and is arbitrarily defined otherwise. A simple example is the spherical, compactly supported kernel with width ε_{MF}, defined as [35]

$$\frac{\mathbf{\Sigma}_o(\boldsymbol{\tau})}{\sigma^2_\xi} \triangleq \begin{cases} 1 - \dfrac{3}{2}\dfrac{\|\boldsymbol{\tau}\|_2}{\varepsilon_{MF}} + \dfrac{1}{2}\left(\dfrac{\|\boldsymbol{\tau}\|_2}{\varepsilon_{MF}}\right)^3, & \text{if } \|\boldsymbol{\tau}\|_2 < \varepsilon_{MF} \\ 0, & \text{if } \|\boldsymbol{\tau}\|_2 \geq \varepsilon_{MF} \end{cases}. \tag{3.22}$$

Of course, across (discrete) time slots, $\xi_S(\mathbf{p}, t)$ inherits whiteness without any technical issue.

We should stress that the above assumptions are made for technical reasons and will be transparent in the subsequent analysis, as long as the mild constraint (3.20) is satisfied; from the perspective of the relays, all evaluations of $\xi_S(\mathbf{p}, t)$, at each time slot, will be independent to each other. And, of course, ε_{MF} may be chosen small enough, such that (3.20) is satisfied virtually always, assuming that the relays are sufficiently far apart from each other, and/or that, at each time slot t, their new positions are relatively close to their old positions, at time slot $t-1$.

Based on the explicit statistical description of $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$ presented above, we now additionally demand that both are spatial fields with *(everywhere) continuous sample paths*. Equivalently, we demand that, for every $\omega \in \Omega$, $\sigma_S(\omega, \mathbf{p}, t) \in \mathsf{C}(\mathcal{S})$ and $\xi_S(\omega, \mathbf{p}, t) \in \mathsf{C}(\mathcal{S})$, where $\mathsf{C}(\mathcal{A})$ denotes the set of continuous functions on some qualifying set $\mathcal{A}$. Sample path continuity of stationary Gaussian fields may be guaranteed under mild conditions on the respective lag-dependent covariance kernel, as the following result suggests, however in a, slightly weaker, *almost everywhere* sense.

Theorem 1. ($a.e.$**-Continuity of Gaussian Fields** [36–38]) *Let $X(\boldsymbol{s})$, $\boldsymbol{s} \in \mathbb{R}^N$, be a real-valued, zero-mean, stationary Gaussian random field with a continuous covariance kernel $\mathbf{\Sigma}_X : \mathbb{R}^N \to \mathbb{R}$. Suppose that there exist constants $0 < c < +\infty$ and $\varepsilon, \zeta > 0$, such that*

$$1 - \frac{\mathbf{\Sigma}_X(\boldsymbol{\tau})}{\mathbf{\Sigma}_X(\mathbf{0})} \leq \frac{c}{\left|\log\left(\|\boldsymbol{\tau}\|_2\right)\right|^{1+\varepsilon}}, \tag{3.23}$$

for all $\boldsymbol{\tau} \in \left\{\boldsymbol{x} \in \mathbb{R}^N \middle| \|\boldsymbol{x}\|_2 < \zeta\right\}$. Then, $X(\boldsymbol{s})$ is $\mathcal{P}$-almost everywhere sample path continuous, or, equivalently, $\mathcal{P} -a.e.$-continuous, on every compact subset $\mathcal{K} \subset \mathbb{R}^N$ and, therefore, on $\mathbb{R}^N$ itself. Additionally, $X(\boldsymbol{s})$ is bounded, $\mathcal{P}$-almost everywhere, as well.

Utilizing Theorem 1 and generically assuming that $\boldsymbol{\Sigma}_{\xi} \triangleq \boldsymbol{\Sigma}_{o}$, it is possible to show that both fields $\sigma_S(\mathbf{p},t)$ and $\xi_S(\mathbf{p},t)$ satisfy the respective conditions and thus, that both fields are *a.e.*-continuous on $\mathcal{S}$. For $\sigma_S(\mathbf{p},t)$, the reader is referred to ([37], Example 2.2). Of course, instead of $\boldsymbol{\Sigma}_{\sigma}$, any other kernel may be considered, as long as the condition Theorem 1 is satisfied.

As far as $\xi_S(\mathbf{p},t)$ is concerned, let us choose $\varepsilon \equiv 1$ and $\zeta \equiv 1$. We thus need to show that, for every $\tau \triangleq \|\boldsymbol{\tau}\|_2 \in [0,1)$, it holds that

$$1-\frac{\boldsymbol{\Sigma}_o(\boldsymbol{\tau})}{\boldsymbol{\Sigma}_o(\mathbf{0})} \le \frac{c}{\left(\log\left(\|\boldsymbol{\tau}\|_2\right)\right)^2}, \tag{3.24}$$

or, equivalently,

$$1-\left(1-\frac{3}{2}\frac{\tau}{\varepsilon_{MF}}+\frac{1}{2}\left(\frac{\tau}{\varepsilon_{MF}}\right)^3\right)\mathbb{1}_{\{\tau<\varepsilon_{MF}\}} \le \frac{c}{\left(\log(\tau)\right)^2}, \tag{3.25}$$

for some finite, positive constant c. We first consider the case where $1 > \tau \ge \varepsilon_{MF} > 0$ (whenever $\varepsilon_{MF} < 1$, of course). We then have

$$1 \le \frac{\left(\log\left(\varepsilon_{MF}\right)\right)^2}{\left(\log(\tau)\right)^2} \triangleq \frac{c_1}{\left(\log(\tau)\right)^2}, \tag{3.26}$$

easily verifying the condition required by Theorem 1. Now, when $0 \le \tau < \min\{\varepsilon_{MF}, 1\}$, it is easy to see that there exists a finite $c_2 > 0$, such that

$$\tau \le \frac{c_2}{\left(\log(\tau)\right)^2}. \tag{3.27}$$

If $\tau \equiv 0$, then the inequality above holds for any choice of c_2. If $\tau > 0$, define a function $h : (0,1) \to \mathbb{R}_+$, as

$$h(\tau) \triangleq \tau\left(\log(\tau)\right)^2. \tag{3.28}$$

By a simple first derivative test, it follows that

$$\begin{aligned} h(\tau) &\le \max_{\tau\in(0,1)} h(\tau) \\ &\equiv h\left(\exp(-2)\right) \\ &\equiv 4\exp(-2), \quad \forall \tau \in (0,1). \end{aligned} \tag{3.29}$$

Consequently, (3.27) is (loosely) satisfied for all $\tau \in [0, \min\{\varepsilon_{MF}, 1\}) \subseteq (0,1)$, by choosing $c_2 \equiv 4\exp(-2)$. Now, observe that

$$\frac{3}{2}\frac{\tau}{\varepsilon_{MF}}-\frac{1}{2}\left(\frac{\tau}{\varepsilon_{MF}}\right)^3 < \frac{3}{2}\frac{\tau}{\varepsilon_{MF}} \le \frac{3c_2}{2\varepsilon_{MF}\left(\log(\tau)\right)^2}. \tag{3.30}$$

Finally, simply choose

$$\begin{aligned} c &\equiv \max\left\{c_1, \frac{3c_2}{2\varepsilon_{MF}}\right\} \\ &\equiv \max\left\{\left(\log\left(\varepsilon_{MF}\right)\right)^2, \frac{6\exp(-2)}{\varepsilon_{MF}}\right\} < +\infty, \end{aligned} \tag{3.31}$$

which immediately implies (3.25). Therefore, we have shown that, if we choose $\boldsymbol{\Sigma}_{\xi} \equiv \boldsymbol{\Sigma}_{o}$, then, for any fixed, but *arbitrarily small* $\varepsilon_{MF} > 0$, the spatial field $\xi_S(\mathbf{p}, t)$ will also be almost everywhere sample path continuous.

Observe that, via the analysis above, sample path continuity of the involved fields can be ascertained, but only in the only almost everywhere sense. Nevertheless, it easy to show that there always exist everywhere sample path continuous fields $\widetilde{\sigma}_S(\mathbf{p}, t)$ and $\widetilde{\xi}_S(\mathbf{p}, t)$, which are *indistinguishable* from $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$, respectively [39]. Therefore, there is absolutely no loss of generality if we take both $\sigma_S(\mathbf{p}, t)$ and $\xi_S(\mathbf{p}, t)$ to be sample path continuous, *everywhere* in Ω, and we will do so, hereafter.

Sample path continuity of all fields $\sigma_{S(D)}(\mathbf{p}, t)$ and $\xi_{S(D)}(\mathbf{p}, t)$ will be essential in Section 4, where we rigorously discuss optimality of the proposed relay motion control framework, with special focus on the relay beamforming problem.

We close this section by discussing, in some more detail, the temporal properties of the *evaluations* of the fields $\sigma_S(\mathbf{p}, t)$ and $\sigma_D(\mathbf{p}, t)$ at any *deterministic* set of N (say) positions $\{\mathbf{p}_i \in \mathcal{S}\}_{i\in\mathbb{N}_N^+}$, *same* across all N_T time slots. This results in the zero-mean, stationary temporal Gaussian process

$$\boldsymbol{C}(t) \triangleq \left[\{\sigma_S(\mathbf{p}_i, t)\}_{i\in\mathbb{N}_N^+} \ \{\sigma_D(\mathbf{p}_i, t)\}_{i\in\mathbb{N}_N^+}\right]^{\boldsymbol{T}} \in \mathbb{R}^{2N\times 1}, \quad t \in \mathbb{N}_{N_T}^+, \tag{3.32}$$

with *matrix covariance kernel* $\boldsymbol{\Sigma}_{\boldsymbol{C}} : \mathbb{Z} \to \mathbb{S}_+^{2N}$, defined, under the specific spatiotemporal model considered, as

$$\boldsymbol{\Sigma}_{\boldsymbol{C}}(\nu) \triangleq \exp\left(-\frac{|\nu|}{\gamma}\right) \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \in \mathbb{S}_+^{2N}, \tag{3.33}$$

where $\nu \triangleq t - s$, for all $(t, s) \in \mathbb{N}_{N_T}^+ \times \mathbb{N}_{N_T}^+$,

$$\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \triangleq \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix} \otimes \widehat{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \in \mathbb{S}_+^{2N}, \tag{3.34}$$

$$\kappa \triangleq \exp\left(-\frac{\|\mathbf{p}_S - \mathbf{p}_D\|_2}{\delta}\right) < 1, \tag{3.35}$$

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{C}}(i, j) \triangleq \boldsymbol{\Sigma}_{\sigma}\left(\mathbf{p}_i - \mathbf{p}_j\right), \quad \forall (i, j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+, \tag{3.36}$$

and with "$\otimes$" denoting the operator of the Kronecker product. Then, the following result is true.

Theorem 2. **($\boldsymbol{C}(t)$ is Markov)** *For any deterministic, time invariant set of points* $\{\mathbf{p}_i \in \mathcal{S}\}_{i\in\mathbb{N}_N^+}$, *the vector process* $\boldsymbol{C}(t) \in \mathbb{R}^{2N\times 1}$, $t \in \mathbb{N}_{N_T}^+$, *as defined in (3.32)-(3.36), may be represented as a stable order-1 vector autoregression, satisfying the linear stochastic difference equation*

$$\boldsymbol{X}(t) \equiv \varphi \boldsymbol{X}(t-1) + \boldsymbol{W}(t), \quad t \in \mathbb{N}_{N_T}^+, \tag{3.37}$$

where

$$\varphi \triangleq \exp(-1/\gamma) < 1, \tag{3.38}$$

$$\boldsymbol{X}(0) \sim \mathcal{N}\left(\mathbf{0}, \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}\right) \quad \textit{and} \tag{3.39}$$

$$\boldsymbol{W}(t) \overset{i.i.d.}{\sim} \mathcal{N}\left(\mathbf{0}, \left(1 - \varphi^2\right) \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}\right), \quad \forall t \in \mathbb{N}_{N_T}^+. \tag{3.40}$$

In particular, $\boldsymbol{C}(t)$ *is Markov.*

Proof of Theorem 2. The proof is a standard exercise in time series; see Appendix A. ■

From a practical point of view, Theorem 2 is extremely valuable. Specifically, the Markovian representation of $\boldsymbol{C}(t)$ may be employed in order to *efficiently simulate* the spatiotemporal paths of the communication channel on any finite, but arbitrarily fine grid. This is important, since it allows detailed *numerical evaluation* of all methods developed in this work. Theorem 2 also reveals that the channel model we have considered actually agrees with experimental results presented in, for instance, [33, 34], which show that autoregressive processes constitute an adequate model for stochastically describing temporal correlations among wireless communication links.

Remark 1. Unfortunately, to the best of our knowledge, the channel process along a specific relay trajectory, presented in Section 3.1, where the positions of the relays are allowed to vary across time slots is no longer stationary and may not be shown to satisfy the Markov Property. Therefore, in our analysis presented hereafter, we regard the aforementioned process as a general, nonstationary Gaussian process. All inference results presented below are based on this generic representation. ■

Remark 2. For simplicity, all motion control problems in this paper are formulated on the plane (some subset of $\mathbb{R}^2$). This means that any motion of the relays of the network along the third dimension of the space is indifferent to our channel model. Nevertheless, under appropriate (based on the requirements discussed above) assumptions concerning 3D wireless channel modeling, all subsequent arguments would hold in exactly the same fashion when fully unconstrained motion in $\mathbb{R}^3$ is assumed to affect the quality of the wireless channel. ■

4 Spatially Controlled Relay Beamforming

In this section, we formulate and solve the spatially controlled relay beamforming problem, advocated in this paper. The beamforming objective adopted will be *maximization of the Signal-to-Interference+Noise Ratio (SINR) at the destination (measuring network QoS)*, under a total power budget at the relays. For the single-source single-destination setting considered herein, the aforementioned beamforming problem admits a closed form solution, a fact which will be important in deriving optimal relay motion control policies, in a tractable fashion. But first, let us present the general scheduling schema of the proposed mobile beamforming system, as well as some technical preliminaries on stochastic programming and optimal control, which will be used repeatedly in the analysis to follow.

4.1 Joint Scheduling of Communications & Controls

At each time slot $t \in \mathbb{N}^+_{N_T}$ and assuming the same carrier for all communication tasks, we employ a basic joint communication/decision making TDMA-like protocol, as follows:

1. The source broadcasts a pilot signal to the relays, which then estimate their respective channels relative to the source.

2. The same procedure is carried out for the channels relative to the destination.

3. Then, based on the estimated CSI, the relays beamform in AF mode (assume perfect CSI estimation).

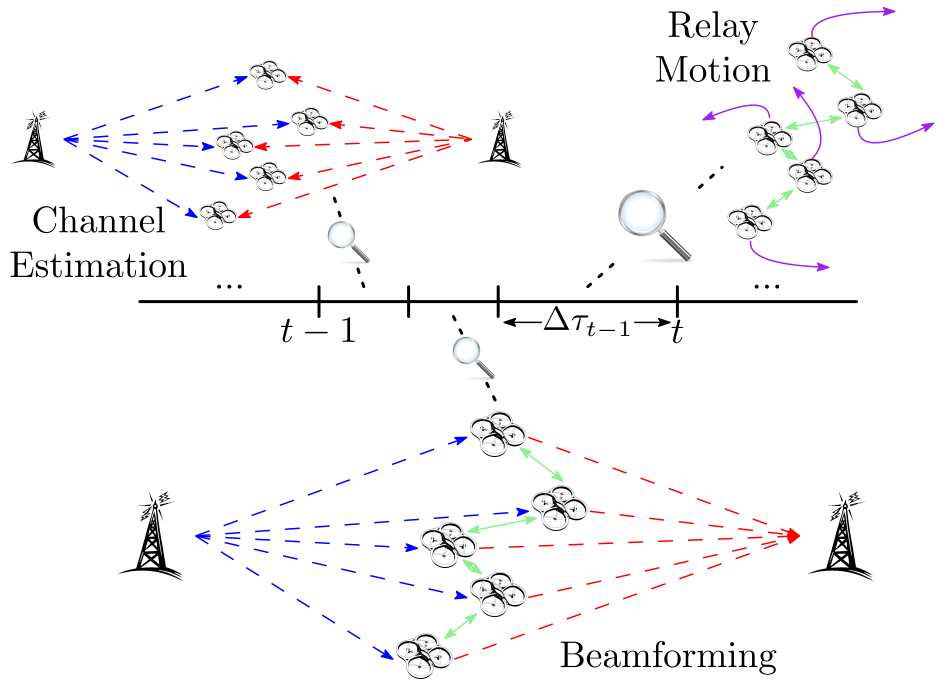

Figure 4.1: Proposed TDMA-like joint scheduling protocol for communications and controls.

4. Based on the CSI received *so far*, strategic decision making is implemented, motion controllers of the relays are determined and relays are steered to their updated positions.

The above sequence of actions is repeated for all N_T time slots, corresponding to the total operational horizon of the system. This simple scheduling protocol is graphically depicted in Fig. 4.1.

Concerning relay kinematics, it is assumed that the relays obey the differential equation

$$\dot{\mathbf{p}}\left(\tau\right) \equiv \mathbf{u}\left(\tau\right), \quad \forall \tau \in \left[0, T\right], \tag{4.1}$$

where $\mathbf{u} \triangleq \left[\mathbf{u}_1 \, \ldots \, \mathbf{u}_R\right]^{\boldsymbol{T}} \in \mathcal{S}^R$, with $\mathbf{u}_i : \left[0, T\right] \rightarrow \mathcal{S}$ being the motion controller of relay $i \in \mathbb{N}_R^+$. Apparently, relay motion is in continuous time. However, assuming the relays may move *only after their controls have been determined and up to the start of the next time slot*, we can write

$$\mathbf{p}\left(t\right) \equiv \mathbf{p}\left(t-1\right) + \int_{\Delta\tau_{t-1}} \mathbf{u}_{t-1}\left(\tau\right) \mathrm{d}\tau, \quad \forall t \in \mathbb{N}_{N_T}^2, \tag{4.2}$$

with $\mathbf{p}\left(1\right) \equiv \mathbf{p}_{init}$, and where $\Delta\tau_t \subset \mathbb{R}$ and $\mathbf{u}_t : \Delta\tau_t \rightarrow \mathcal{S}^R$ denote the time interval that the relays are allowed to move in and the respective relay controller, in each time slot $t \in \mathbb{N}_{N_T-1}^+$. It holds that $\mathbf{u}\left(\tau\right) \equiv \sum_{t \in \mathbb{N}_{N_T-1}^+} \mathbf{u}_t\left(\tau\right) \mathbb{1}_{\Delta\tau_t}\left(\tau\right)$, *where* τ *belongs in the first* $N_T - 1$ time slots.Of course, at each time slot t, the length of $\Delta\tau_t$, $|\Delta\tau_t|$, must be sufficiently small such that the temporal correlations of the CSI at adjacent time slots are sufficiently strong. These correlations are controlled by the correlation time parameter γ, which can be a function of the slot width. Therefore, the velocity of the relays must be of the order of $\left(|\Delta\tau_t|\right)^{-1}$. In this work, though, we assume that the relays are not explicitly resource constrained, in terms of their motion.

Now, regarding the form of the relay motion controllers $\mathbf{u}_{t-1}\left(\tau\right), \tau \in \Delta\tau_{t-1}$, *given a goal position vector at time slot* t, $\mathbf{p}^o\left(t\right)$, it suffices to fix a path in $\mathcal{S}^R$, such that the points $\mathbf{p}^o\left(t\right)$ and $\mathbf{p}\left(t-1\right)$ are connected in at most time $|\Delta\tau_{t-1}|$. A generic choice for such a path is the straight line[1] connecting $\mathbf{p}_i^o(t)$ and $\mathbf{p}_i(t-1)$, for all $i \in \mathbb{N}_R^+$. Therefore, we may choose the relay controllers at time slot $t-1 \in \mathbb{N}_{N_T-1}^+$ as

$$\mathbf{u}_{t-1}^o(\tau) \triangleq \frac{1}{\Delta\tau_{t-1}}\left(\mathbf{p}^o(t) - \mathbf{p}(t-1)\right), \quad \forall \tau \in \Delta\tau_{t-1}. \tag{4.3}$$

As a result, any motion control problem considered hereafter can now be formulated in terms of specifying the goal relay positions at the next time slot, given their positions at the current time slot (and the observed CSI).

In the following, let $\mathscr{C}(\mathcal{T}_t)$ denote the set of channel gains observed by the relays, *along the paths of their point trajectories* $\mathcal{T}_t \triangleq \{\mathbf{p}(1)\ldots\mathbf{p}(t)\}$, $t \in \mathbb{N}_{N_T}^+$. Then, $\mathcal{T}_t$ may be recursively updated as $\mathcal{T}_t \equiv \mathcal{T}_{t-1} \cup \{\mathbf{p}(t)\}$, for all $t \in \mathbb{N}_{N_T}^+$, with $\mathcal{T}_0 \triangleq \varnothing$. In a technically precise sense, $\{\mathscr{C}(\mathcal{T}_t)\}_{t\in\mathbb{N}_{N_T}^+}$ will also denote the filtration generated by the CSI observed at the relays, *along* $\mathcal{T}_t$, interchangeably. In other words, in case the trajectories of the relays are themselves random, then $\mathscr{C}(\mathcal{T}_t)$ denotes the σ-algebra generated by *both* the CSI observed up to and including time slot t *and* $\mathbf{p}(1)\ldots\mathbf{p}(t)$, for all $t \in \mathbb{N}_{N_T}^+$. Additionally, we define $\mathscr{C}(\mathcal{T}_0) \equiv \mathscr{C}(\{\varnothing\})$ as $\mathscr{C}(\mathcal{T}_0) \triangleq \{\varnothing, \Omega\}$, that is, as the trivial σ-algebra, and we may occasionally refer to time $t \equiv 0$, as a *dummy time slot*, by convention.

4.2 2-Stage Stochastic Optimization of Beamforming Weights and Relay Positions: Base Formulation & Methodology

At each time slot $t \in \mathbb{N}_{N_T}^+$, given the current CSI encoded in $\mathscr{C}(\mathcal{T}_t)$, we are interested in determining $\boldsymbol{w}^o(t) \triangleq [w_1(t)\; w_2(t)\;\ldots\; w_R(t)]^{\boldsymbol{T}}$, as an optimal solution to a beamforming optimization problem, as a functional of $\mathscr{C}(\mathcal{T}_t)$. Let the *optimal value* (say infimum) of this problem be the process $V_t \equiv V(\mathbf{p}(t), t)$, a functional of the CSI encoded in $\mathscr{C}(\mathcal{T}_t)$, depending on the positions of the relays at time slot t.

Suppose that, *at time slot* $t-1$, an oracle reveals $\mathscr{C}(\mathcal{T}_t \equiv \mathcal{T}_{t-1} \cup \{\mathbf{p}(t)\})$, which also determines the channels corresponding to the new positions of the relays at the next time slot t. Then, we could further consider optimizing V_t with respect to $\mathbf{p}(t)$, representing the new position of the relays. But note that, $\mathscr{C}(\mathcal{T}_t)$ is not physically observable and in the absence of the oracle, optimizing V_t with respect to $\mathbf{p}(t)$ is impossible, since, given $\mathscr{C}(\mathcal{T}_{t-1})$, the channels at any position of the relays are nontrivial random variables. However, it is reasonable to search for the best decision on the positions of the relays at time slot t, as a functional of the available information encoded in $\mathscr{C}(\mathcal{T}_{t-1})$, such that V_t is optimized *on average*. This procedure may be formally formulated as a 2-*stage stochastic program* [26],

$$\begin{array}{rl} \underset{\mathbf{p}(t)}{\text{minimize}} & \mathbb{E}\{V(\mathbf{p}(t), t)\} \\ \text{subject to} & \mathbf{p}(t) \equiv \mathcal{M}(\mathscr{C}(\mathcal{T}_{t-1})) \in \mathcal{C}(\mathbf{p}^o(t-1)), \\ & \text{for some } \mathcal{M} : \mathbb{R}^{4R(t-1)} \to \mathbb{R}^{2R} \end{array}, \tag{4.4}$$

to be solved at each $t-1 \in \mathbb{N}_{N_T-1}^+$, where $\mathcal{C} : \mathbb{R}^{2R} \rightrightarrows \mathbb{R}^{2R}$ is a multifunction, with $\mathcal{C}(\mathbf{p}^o(t-1)) \subseteq \mathcal{S}^R$ representing a physically feasible spatial neighborhood around the point $\mathbf{p}^o(t-1) \in \mathcal{S}^R$, the decision vector *selected* at time $t-2 \in \mathbb{N}_{N_T-2}$ (recall that $t \equiv 0$ denotes a dummy time slot). Note that, in

[1] Caution is needed here, due to the possibility of physical collisions among relays themselves, or among relays and other physical obstacles in the workspace, $\mathcal{S}$. Nevertheless, for simplicity, we assume that either such events never occur, or that, if they do, there exists some transparent collision avoidance mechanism implemented at each relay, which is out of our direct control.

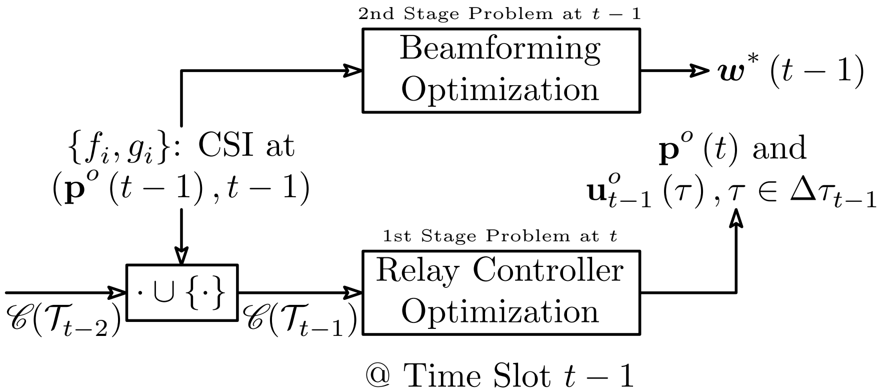

Figure 4.2: 2-Stage optimization of beamforming weights and spatial relay controllers. The variables $\boldsymbol{w}^o(t-1)$, $\mathbf{u}^o_{t-1}$ and $\mathbf{p}^o(t)$ denote the optimal beamforming weights and relay controllers at time slot $t-1$, and the optimal relay positions at time slot t, respectively.

general, the decision *selected* at $t-2$, $\mathbf{p}^o(t-1)$, may not be an optimal decision for the respective problem solved at $t-2$ and implemented at $t-1$. To distinguish $\mathbf{p}^o(t-1)$ from an optimal decision at $t-2$, the latter will be denoted as $\mathbf{p}^*(t-1)$, for all $t \in \mathbb{N}^2_{N_T}$. Also note that, in order for (4.4) to be well defined, important technical issues, such as measurability of V_t and existence of its expectation at least for each feasible decision $\mathbf{p}(t)$, should be precisely resolved. Problem (4.5), together with the respective beamforming problem with optimal value V_t (which will focus on shortly) are referred to as the *first-stage problem* and the *second-stage problem*, respectively [26]. Hereafter, aligned with the literature, any feasible choice for the decision variable $\mathbf{p}(t)$ in (4.5), will be interchangeably called an *(admissible) policy*. A generic block representation of the proposed 2-stage stochastic programming approach is depicted in Fig. 4.2.

Mainly due to the arbitrary structure of the function $\mathcal{M}$, (4.4) is too general to consider, within a reasonable analytical framework. Thus, let us slightly constrain the decision set of (4.4) to include *only measurable decisions*, resulting in the formulation

$$\begin{array}{ll} \underset{\mathbf{p}(t)}{\text{minimize}} & \mathbb{E}\{V(\mathbf{p}(t), t)\} \\ \text{subject to} & \mathbf{p}(t) \equiv \mathcal{M}(\mathscr{C}(\mathcal{T}_{t-1})) \in \mathcal{C}(\mathbf{p}^o(t-1)), \\ & \mathcal{M}^{-1}(\mathcal{A}) \in \mathscr{B}\left(\mathbb{R}^{4R(t-1)}\right), \forall \mathcal{A} \in \mathscr{B}\left(\mathbb{R}^{2R}\right) \end{array}, \tag{4.5}$$

provided, of course, that the stochastic program (4.5) is well defined. The second constraint in (4.5) is equivalent to $\mathcal{M}$ being Borel measurable, instead of being any arbitrary function, as in (4.4).

Provided its well definiteness, the stochastic program (4.5) is difficult to solve, most importantly because of its *variational character*; the decision variable $\mathbf{p}(t)$ is constrained to be a functional of the CSI observed up to and including time $t-1$. A very powerful tool, which will enable us to both make (4.5) meaningful and overcome the aforementioned difficulty, is the *Fundamental Lemma of Stochastic Control* [23–28], which in fact refers to a family of technical results related to the *interchangeability* of integration (expectation) and minimization in general stochastic programming. Under the framework of the Fundamental Lemma, in Appendix B, we present a detailed discussion, best suited for the purposes of this paper, which is related to the important technical issues, arising when one wishes to meaningfully define and tractably simplify "hard", variational problems of the form of (4.5).

In particular, Lemma 4, presented in Section 8.2.4 (Appendix B), identifies six sufficient technical conditions (conditions **C1-C6**, see statement of Lemma 4), under which the variational problem (4.5) is *exchangeable* by the structurally simpler, *pointwise* optimization problem

$$\begin{array}{ll} \underset{\mathbf{p}(t)}{\text{minimize}} & \mathbb{E}\left\{V\left(\mathbf{p}\left(t\right),t\right)|\mathscr{C}\left(\mathcal{T}_{t-1}\right)\right\} \\ \text{subject to} & \mathbf{p}\left(t\right)\in\mathcal{C}\left(\mathbf{p}^{o}\left(t-1\right)\right) \end{array}, \tag{4.6}$$

to be solved at each $t-1\in\mathbb{N}^{+}_{N_T-1}$. Observe that, in (4.6), the decision variable $\mathbf{p}\left(t\right)$ is constant, as opposed to (4.5), where the decision variable $\mathbf{p}\left(t\right)$ is itself a functional of the observed information at time slot $t-1$, that is, a policy. Provided that CSI $\mathscr{C}\left(\mathcal{T}_{t-1}\right)$ and $\mathbf{p}^{o}\left(t-1\right)$ are known and that the involved conditional expectation can be somehow evaluated, (4.6) constitutes an ordinary, nonlinear optimization problem.

If Lemma 4 is in power, *exchangeability* of (4.5) by (4.6) is understood in the sense that the optimal value of (4.5), which is a number, coincides with the *expectation* of optimal value of (4.6), which turns out to be a measurable function of $\mathscr{C}\left(\mathcal{T}_{t-1}\right)$. In other words, minimization is *interchangeable* with integration, in the sense that

$$\inf_{\mathbf{p}(t)\in\mathcal{D}_t}\mathbb{E}\left\{V\left(\mathbf{p}\left(t\right),t\right)\right\}\equiv\mathbb{E}\left\{\inf_{\mathbf{p}(t)\in\mathcal{C}\left(\mathbf{p}^{o}(t-1)\right)}\mathbb{E}\left\{V\left(\mathbf{p}\left(t\right),t\right)|\mathscr{C}\left(\mathcal{T}_{t-1}\right)\right\}\right\}, \tag{4.7}$$

for all $t\in\mathbb{N}^{2}_{N_T}$, where $\mathcal{D}_t$ denotes the set of feasible decisions for (4.5). What is more, under the aforementioned technical conditions of Lemma 4, exchangeability implies that, if there exists an admissible policy of (4.5), say $\mathbf{p}^{*}\left(t\right)$, which solves (4.6), then $\mathbf{p}^{*}\left(t\right)$ is also optimal for (4.5). Additionally, Lemma 4 implies existence of at least one optimal solution to (4.6), which is simultaneously feasible and, thus, optimal, for the original stochastic program (4.5). If, further, (4.6) features a unique optimal solution, say $\mathbf{p}^{*}\left(t\right)$, then $\mathbf{p}^{*}\left(t\right)$ must be an optimal solution to (4.5).

In the next subsection, we will specify the optimal value of the second-stage subproblem, V_t, for each time $t\in\mathbb{N}^{+}_{N_T}$. That is, we will consider a fixed criterion for implementing relay beamforming (recourse actions) at each t, *after* the predictive decisions on the positions of the relays have been made (*at time* $t-1$) and the relays have moved to their new positions, implying that the CSI at time at time t has been revealed. Of course, one of the involved challenges will be to explicitly show that Conditions **C1-C6** are satisfied for each case considered, so that we can focus on solving the ordinary nonlinear optimization problem (4.6), instead of the much more difficult variational problem (4.5). The other challenge we will face is actually solving (4.6).

Remark 3. It would be important to note that the pointwise problem (4.5) admits a reasonable and intuitive interpretation: At each time slot $t-1$, instead of (deterministically) optimizing V_t with respect to $\mathbf{p}\left(t\right)$ in $\mathcal{C}\left(\mathbf{p}^{o}\left(t-1\right)\right)$, which is, of course, impossible, one considers optimizing *a projection of* $V\left(\mathbf{p},t\right)$*,* $\mathbf{p}\in\mathcal{S}^{R}$ *onto the space of all measurable functionals of* $\mathscr{C}\left(\mathcal{T}_{t-1}\right)$, which corresponds to the information observed by the relays, up to $t-1$. Provided that, for every $\mathbf{p}\in\mathcal{S}^{R}$, $V\left(\mathbf{p},t\right)$ is in the Hilbert space of square-integrable, real-valued functions relative to $\mathcal{P}$, $\mathcal{L}_2\left(\Omega,\mathscr{F},\mathcal{P};\mathbb{R}\right)$, it is then reasonable to consider orthogonal projections, that is, the Minimum Mean Square Error (MMSE) estimate, or, more accurately, prediction of $V\left(\mathbf{p},t\right)$ given $\mathscr{C}\left(\mathcal{T}_{t-1}\right)$. This, of course, coincides with the conditional expectation $\mathbb{E}\left\{V\left(\mathbf{p},t\right)|\mathscr{C}\left(\mathcal{T}_{t-1}\right)\right\}$. One then optimizes the *random utility* $\mathbb{E}\left\{V\left(\mathbf{p},t\right)|\mathscr{C}\left(\mathcal{T}_{t-1}\right)\right\}$, with respect to $\mathbf{p}$ in the random set $\mathcal{C}\left(\mathbf{p}^{o}\left(t-1\right)\right)$, as in (4.6).

Although there is nothing technically wrong with actually starting with (4.6) as our *initial* problem formulation, and essentially bypassing the technical difficulties of (4.5), the fact that the objective of (4.6) depends on $\mathscr{C}(\mathcal{T}_{t-1})$ does not render it a useful optimality criterion. This is because the objective of (4.6) quantifies the performance of a *single* decision, *only conditioned* on $\mathscr{C}(\mathcal{T}_{t-1})$, *despite* the fact that an optimal solution to (4.6) (provided it exists) constitutes itself a functional of $\mathscr{C}(\mathcal{T}_{t-1})$. In other words, the objective of (4.6) *does not* quantify the performance of *a policy (a decision rule)*; in order to do that, any reasonable performance criterion should assign *a number* to each policy, ranking its quality, and *not* a function depending on $\mathscr{C}(\mathcal{T}_{t-1})$. The expected utility $\mathbb{E}\{V_t\}$ of the variational problem (4.5) constitutes a suitable such criterion. And by the Fundamental Lemma, (4.5) may be indeed reduced to (4.6), which can thus be regarded as a proxy for solving the former.

There are two main reasons justifying our interest in policies, rather than individual decisions. First, one should be interested in the *long-term behavior* of the beamforming (in our case) system, in the sense that it should be possible to assess system performance if the system is used repeatedly over time, e.g., periodically (every hour, day) or on demand. For example, consider a beamforming system (the "experiment"), which operates for N_T time slots and dependently *restarts* its operation at time slots kN_T+1, for k in some subset of $\mathbb{N}^+$. This might be practically essential for maintaining system stability over time, saving on resources, etc. It is then clear that merely quantifying the performance of individual decisions is meaningless, from an operational point of view; simply, the random utility approach quantifies performance *only along a specific path of the observed information*, $\mathscr{C}(\mathcal{T}_{t-1})$, for $t \in \mathbb{N}^+_{N_T}$. This issue is more profound when channel observations taking specific values correspond to events of zero measure (this is actually the case with the Gaussian channel model introduced in Section 3). On the contrary, it is of interest to jointly quantify system performance when decisions are made for different outcomes of the sample space Ω. This immediately results in the need for quantifying the performance of different policies (decision rules), and this is only possible by considering variational optimization problems, such as (4.5).

Additionally, because decisions are made *in stages*, it is of great interest to consider how the system performs *across time slots*, or, in other words, to discover *temporal trends* in performance, if such trends exist. In particular, for the beamforming problem considered in this paper, we will be able to theoretically characterize system behavior under both suboptimal and optimal decision making, in the average (expected) sense (see Section 4.4), *across all time slots*; this is impossible to do for each possible outcome of the sample space, individually, when the random utility approach is considered.

The second main reason for considering the variational program (4.5) as our main objective, instead of (4.6), is practical, and extremely important from an engineering point of view. The expected utility approach assigns, at each time slot, a number to each policy, quantifying its quality. Simulating repeatedly the system and invoking the Law of Large Numbers, one may obtain excellent estimates of the expected performance of the system, quantified by the chosen utility. Therefore, the systematic experimental assessment of a particular sequence of policies (one for each time slot) is readily possible. Apparently, such experimental validation approach is impossible to perform by adopting the random (conditional) utility approach, since the performance of the system will be quantified via a real valued (in general) random quantity. ■

Remark 4. The stochastic programming methodology presented in this subsection is very general and can support lots of choices in regard to the structure of the second-stage subproblem, V_t. As shown in the discussion developed in Appendix B, the key to showing the validity of the Fundamental Lemma is the set of conditions **C1-C6**. If these are satisfied, it is then possible to convert the original, variational problem into a pointwise one, while strictly preserving optimality. ■

4.3 SINR Maximization at the Destination

The basic and fundamentally important beamforming criterion considered in this paper is that of enhancing network QoS, or, in other words, maximizing the respective SINR at the destination, subject to a total power budget at the relays. At each time slot $t \in \mathbb{N}^{+}_{N_T}$, *given* CSI encoded in $\mathscr{C}(\mathcal{T}_t)$ and with $\boldsymbol{w}(t) \triangleq [w_1(t) \ \ldots \ w_R(t)]^{\boldsymbol{T}}$, this may be achieved by formulating the constrained optimization problem [1,4]

$$\begin{array}{ll} \underset{\boldsymbol{w}(t)}{\text{maximize}} & \dfrac{\mathbb{E}\{P_S(t)|\mathscr{C}(\mathcal{T}_t)\}}{\mathbb{E}\{P_{I+N}(t)|\mathscr{C}(\mathcal{T}_t)\}} \\ \text{subject to} & \mathbb{E}\{P_R(t)|\mathscr{C}(\mathcal{T}_t)\} \leq P_c \end{array}, \tag{4.8}$$

where $P_R(t)$, $P_S(t)$ and $P_{I+N}(t)$ denote the random instantaneous power at the relays, that of the signal component and that of the interference plus noise component at the destination (see (2.2)), respectively and where $P_c > 0$ denotes the total available relay transmission power. Using the mutual independence assumptions regarding CSI related to the source and destination, respectively, (4.8) can be reexpressed analytically as [1]

$$\begin{array}{ll} \underset{\boldsymbol{w}(t)}{\text{maximize}} & \dfrac{\boldsymbol{w}^{\boldsymbol{H}}(t)\,\mathbf{R}(\mathbf{p}(t),t)\,\boldsymbol{w}(t)}{\sigma_D^2 + \boldsymbol{w}^{\boldsymbol{H}}(t)\,\mathbf{Q}(\mathbf{p}(t),t)\,\boldsymbol{w}(t)} \\ \text{subject to} & \boldsymbol{w}^{\boldsymbol{H}}(t)\,\mathbf{D}(\mathbf{p}(t),t)\,\boldsymbol{w}(t) \leq P_c \end{array}, \tag{4.9}$$

where, dropping the dependence on $(\mathbf{p}(t),t)$ or t for brevity,

$$\mathbf{D} \triangleq P_0 \text{diag}\left(\left[|f_1|^2 \ |f_2|^2 \ \ldots \ |f_R|^2\right]^{\boldsymbol{T}}\right) + \sigma^2 \mathbf{I}_R \in \mathbb{S}^R_{++}, \tag{4.10}$$

$$\mathbf{R} \triangleq P_0 \mathbf{h}\mathbf{h}^{\boldsymbol{H}} \in \mathbb{S}^R_{+}, \text{ with } \mathbf{h} \triangleq [f_1 g_1 \ f_2 g_2 \ \ldots \ f_R g_R]^{\boldsymbol{T}} \text{ and} \tag{4.11}$$

$$\mathbf{Q} \triangleq \sigma^2 \text{diag}\left(\left[|g_1|^2 \ |g_2|^2 \ \ldots \ |g_R|^2\right]^{\boldsymbol{T}}\right) \in \mathbb{S}^R_{++}. \tag{4.12}$$

Note that the program (4.9) is *always feasible, as long as P_c is nonnegative.* It is well known that the optimal value of (4.9) can be expressed in closed form as [1]

$$V_t \equiv V(\mathbf{p}(t),t) \triangleq P_c \lambda_{max}\left(\left(\sigma_D^2 \mathbf{I}_R + P_c \mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2}\right)^{-1} \mathbf{D}^{-1/2}\mathbf{R}\mathbf{D}^{-1/2}\right), \tag{4.13}$$

for all $t \in \mathbb{N}^{+}_{N_T}$. Exploitting the structure of the matrices involved, V_t may also be expressed *analytically* as [4]

$$\begin{aligned} V_t &\equiv \sum_{i \in \mathbb{N}^+_R} \frac{P_c P_0 |f(\mathbf{p}_i(t),t)|^2 |g(\mathbf{p}_i(t),t)|^2}{P_0 \sigma_D^2 |f(\mathbf{p}_i(t),t)|^2 + P_c \sigma^2 |g(\mathbf{p}_i(t),t)|^2 + \sigma^2 \sigma_D^2} \\ &\triangleq \sum_{i \in \mathbb{N}^+_R} V_I(\mathbf{p}_i(t),t), \quad \forall t \in \mathbb{N}^{+}_{N_T}. \end{aligned} \tag{4.14}$$

Adopting the 2-stage stochastic optimization framework presented and discussed in Section 4.2, we are now interested, at each time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, in the program

$$\begin{array}{ll} \underset{\mathbf{p}(t)}{\text{maximize}} & \mathbb{E}\left\{ \sum_{i \in \mathbb{N}^+_R} V_I\left(\mathbf{p}_i\left(t\right), t\right) \right\} \\ \text{subject to} & \mathbf{p}\left(t\right) \equiv \mathcal{M}\left(\mathscr{C}\left(\mathcal{T}_{t-1}\right)\right) \in \mathcal{C}\left(\mathbf{p}^o\left(t-1\right)\right), \\ & \mathcal{M}^{-1}\left(\mathcal{A}\right) \in \mathscr{B}\left(\mathbb{R}^{4R(t-1)}\right), \forall \mathcal{A} \in \mathscr{B}\left(\mathbb{R}^{2R}\right) \end{array}, \tag{4.15}$$

where $\mathbf{p}^o\left(1\right) \in \mathcal{S}^R$ is a known constant, representing the initial positions of the relays. But in order to be able to formulate (4.15) in a well defined manner fully and and simplify it by exploitting the Fundamental Lemma, we have to explicitly verify Conditions **C1-C6** of Lemma 4 in Section 8.2.4 of Appendix B. To this end, let us present a definition.

Definition 1. (Translated Multifunctions) Given nonempty sets $\mathcal{H} \subset \mathbb{R}^N$, $\mathcal{A} \subseteq \mathbb{R}^N$ and any fixed $\boldsymbol{h} \in \mathcal{H}$, $\mathcal{D} : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is called the $(\mathcal{H}, \boldsymbol{h})$-translated multifunction in $\mathcal{A}$, if and only if $\mathcal{D}\left(\boldsymbol{y}\right) \triangleq \left\{\boldsymbol{x} \in \mathcal{A} \middle| \boldsymbol{x} - \boldsymbol{y} \in \mathcal{H}\right\}$, for all $\boldsymbol{y} \in \mathcal{A} - \boldsymbol{h} \triangleq \left\{\boldsymbol{x} \in \mathbb{R}^N \middle| \boldsymbol{x} + \boldsymbol{h} \in \mathcal{A}\right\}$.

Note that translated multifunctions, in the sense of Definition 1, are always unique and nonempty, whenever $\boldsymbol{y} \in \mathcal{A} - \boldsymbol{h}$. We also observe that, if $\boldsymbol{y} \notin \mathcal{A} - \boldsymbol{h}$, $\mathcal{D}\left(\boldsymbol{y}\right)$ is undefined; in fact, outside $\mathcal{A} - \boldsymbol{h}$, $\mathcal{D}$ may be defined arbitrarily, and this will be irrelevant in our analysis. The following assumption on the structure of the compact-valued multifunction $\mathcal{C} : \mathbb{R}^{2R} \rightrightarrows \mathbb{R}^{2R}$ is adopted hereafter, and for the rest of this paper.

Assumption 2. ($\mathcal{C}$ is Translated) *Given any arbitrary compact set $\mathbf{0} \in \mathcal{G}$, $\mathcal{C}$ constitutes the corresponding $(\mathcal{G}, \mathbf{0})$-translated, compact-valued multifunction in $\mathcal{S}^R$.*

Then, the following important result is true.

Theorem 3. (Verification Theorem / SINR Maximization) *Suppose that, at time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, the selected decision at $t-2$, $\mathbf{p}^o\left(t-1\right) \equiv \mathbf{p}^o\left(\omega, t-1\right)$, is measurable relative to $\mathscr{C}\left(\mathcal{T}_{t-2}\right)$. Then, the stochastic program (4.15) satisfies conditions* **C1-C6** *and the Fundamental Lemma applies (see Appendix B, Section 8.2.4, Lemma 4). Additionally, as long as the pointwise program*

$$\begin{array}{ll} \underset{\mathbf{p}}{\text{maximize}} & \sum_{i \in \mathbb{N}^+_R} \mathbb{E}\left\{ V_I\left(\mathbf{p}_i, t\right) \middle| \mathscr{C}\left(\mathcal{T}_{t-1}\right)\right\} \\ \text{subject to} & \mathbf{p} \in \mathcal{C}\left(\mathbf{p}^o\left(t-1\right)\right) \end{array} \tag{4.16}$$

has a unique maximizer $\mathbf{p}^*\left(t\right)$, and $\mathbf{p}^o\left(t\right) \equiv \mathbf{p}^*\left(t\right)$, then $\mathbf{p}^o\left(t\right)$ is $\mathscr{C}\left(\mathcal{T}_{t-1}\right)$-measurable and the condition of the theorem is automatically satisfied at time slot t.

Proof of Theorem 3. See Appendix C. ∎

As Theorem 3 suggests, in order for conditions **C1-C6** to be simultaneously satisfied for all $t \in \mathbb{N}^2_{N_T}$, it is sufficient that the program (4.16) has a *unique* optimal solution, for each t. Although, in general, such requirement might not be particularly appealing, for the problems of interest in this paper, the event where (4.16) does not have a unique optimizer is extremely rare, almost never occurring in practice. Nevertheless, uniqueness of the optimal solution to (4.16) does not constitute a necessary condition for $\mathscr{C}(\mathcal{T}_{t-1})$-measurability of the optimal decision at time slot $t-1$. For instance, $\mathbf{p}^*(t)$ will always be $\mathscr{C}(\mathcal{T}_{t-1})$-measurable when the compact-valued, closed multifunction $\mathcal{C}:\mathbb{R}^{2R}\rightrightarrows\mathbb{R}^{2R}$ is additionally *finite-valued*, and $\mathbf{p}^o(t)\equiv\mathbf{p}^*(t)$. This choice for $\mathcal{C}$ is particularly useful for practical implementations. In any case, as long as conditions **C1-C6** are guaranteed to be satisfied, we may focus exclusively on the pointwise program (4.16), whose expected optimal value, via the Fundamental Lemma, coincides with the optimal value of the original problem (4.15).

By definition, we readily observe that the problem (4.16) is separable. In fact, given that, *for each* $t\in\mathbb{N}^+_{N_T-1}$*, decisions taken and CSI collected so far are available to all relays*, (4.16) can be solved in a *completely distributed fashion at the relays*, with the i-th relay being responsible for solving the program

$$\begin{array}{ll}\underset{\mathbf{p}}{\text{maximize}} & \mathbb{E}\left\{\left.V_I(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}\\ \text{subject to} & \mathbf{p}\in\mathcal{C}_i\left(\mathbf{p}^o(t-1)\right)\end{array},\tag{4.17}$$

at each $t-1\in\mathbb{N}^+_{N_T-1}$, where $\mathcal{C}_i:\mathbb{R}^2\rightrightarrows\mathbb{R}^2$ denotes the corresponding part of $\mathcal{C}$, for each $i\in\mathbb{N}^+_R$. Note that no local exchange of intermediate results is required among relays; given the available information, each relay independently solves its own subproblem. It is also evident that apart from the obvious difference in the feasible set, the optimization problems at each of the relays are identical. The problem, however, with (4.17) is that its objective involves the evaluation of a conditional expectation of a well defined ratio of almost surely positive random variables, which is *impossible to perform analytically*. For this reason, it is imperative to resort to the development of well behaved approximations to (4.17), which, at the same time, would facilitate implementation. In the following, we present two such heuristic approaches.

4.3.1 Approximation by the Method of Statistical Differentials

The first idea we are going to explore is that of approximating the objective of (4.17) by truncated Taylor expansions. Observe that V_I can be equivalently expressed as

$$V_I(\mathbf{p},t)\equiv\frac{1}{\dfrac{\sigma_D^2}{P_c}|g(\mathbf{p},t)|^{-2}+\dfrac{\sigma^2}{P_0}|f(\mathbf{p},t)|^{-2}+\dfrac{\sigma^2\sigma_D^2}{P_cP_0}|f(\mathbf{p},t)|^{-2}|g(\mathbf{p},t)|^{-2}}\triangleq\frac{1}{V_{II}(\mathbf{p},t)},\tag{4.18}$$

for all $(\mathbf{p},t)\in\mathcal{S}\times\mathbb{N}^+_{N_T}$. Then, for $t\in\mathbb{N}^2_{N_T}$, we may locally approximate $\mathbb{E}\left\{\left.V_I(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}$ around the point $\mathbb{E}\left\{\left.V_{II}(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}$ (see Section 3.14.2 in [21]; also known as the *Method of Statistical Differentials*) via a first order Taylor expansion as

$$\mathbb{E}\left\{\left.V_I(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}\approx\frac{1}{\mathbb{E}\left\{\left.V_{II}(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}},\tag{4.19}$$

or via a second order Taylor expansion as

$$\mathbb{E}\left\{\left.V_I(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}\approx\frac{\mathbb{E}\left\{\left.(V_{II}(\mathbf{p},t))^2\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}}{\left(\mathbb{E}\left\{\left.V_{II}(\mathbf{p},t)\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}\right)^3},\tag{4.20}$$

where it is straightforward to show that the square on the numerator can be expanded as

$$(V_{II}(\mathbf{p},t))^2\equiv\left(\frac{\sigma^2\sigma_D^2}{P_cP_0}\right)^2|f(\mathbf{p},t)|^{-4}|g(\mathbf{p},t)|^{-4}+2\frac{\sigma^2\sigma_D^2}{P_cP_0}|f(\mathbf{p},t)|^{-2}|g(\mathbf{p},t)|^{-2}$$

$$+ 2\left(\frac{\sigma^2}{P_0}\right)^2 \frac{\sigma_D^2}{P_c} |f(\mathbf{p},t)|^{-4} |g(\mathbf{p},t)|^{-2} + 2\frac{\sigma^2}{P_0}\left(\frac{\sigma_D^2}{P_c}\right)^2 |f(\mathbf{p},t)|^{-2} |g(\mathbf{p},t)|^{-4}$$
$$+ \left(\frac{\sigma^2}{P_0}\right)^2 |f(\mathbf{p},t)|^{-4} + \left(\frac{\sigma_D^2}{P_c}\right)^2 |g(\mathbf{p},t)|^{-4}. \tag{4.21}$$

The approximate formula (4.20) may be in fact computed in closed form at any point $\mathbf{p} \in \mathcal{S}$, thanks to the following technical, but simple, result.

Lemma 2. (Big Expectations) *Under the wireless channel model introduced in Section 3, it is true that, at any* $\mathbf{p} \in \mathcal{S}$,

$$[F(\mathbf{p},t)\; G(\mathbf{p},t)]^{\boldsymbol{T}} \Big| \mathscr{C}(\mathcal{T}_{t-1}) \sim \mathcal{N}\left(\boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p}), \boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p})\right), \tag{4.22}$$

for all $t \in \mathbb{N}^2_{N_T}$*, and where we define*

$$\boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p}) \triangleq \left[\mu^F_{t|t-1}(\mathbf{p})\; \mu^G_{t|t-1}(\mathbf{p})\right]^{\boldsymbol{T}}, \tag{4.23}$$
$$\mu^F_{t|t-1}(\mathbf{p}) \triangleq \alpha_S(\mathbf{p})\,\ell + \boldsymbol{c}^F_{1:t-1}(\mathbf{p})\,\boldsymbol{\Sigma}^{-1}_{1:t-1}(\boldsymbol{m}_{1:t-1} - \boldsymbol{\mu}_{1:t-1}) \in \mathbb{R}, \tag{4.24}$$
$$\mu^G_{t|t-1}(\mathbf{p}) \triangleq \alpha_D(\mathbf{p})\,\ell + \boldsymbol{c}^G_{1:t-1}(\mathbf{p})\,\boldsymbol{\Sigma}^{-1}_{1:t-1}(\boldsymbol{m}_{1:t-1} - \boldsymbol{\mu}_{1:t-1}) \in \mathbb{R} \quad \textit{and} \tag{4.25}$$
$$\boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p}) \triangleq \begin{bmatrix} \eta^2 + \sigma_\xi^2 & \eta^2 e^{-\frac{\|\mathbf{p}_S - \mathbf{p}_D\|_2}{\delta}} \\ \eta^2 e^{-\frac{\|\mathbf{p}_S - \mathbf{p}_D\|_2}{\delta}} & \eta^2 + \sigma_\xi^2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{c}^F_{1:t-1}(\mathbf{p}) \\ \boldsymbol{c}^G_{1:t-1}(\mathbf{p}) \end{bmatrix} \boldsymbol{\Sigma}^{-1}_{1:t-1} \begin{bmatrix} \boldsymbol{c}^F_{1:t-1}(\mathbf{p}) \\ \boldsymbol{c}^G_{1:t-1}(\mathbf{p}) \end{bmatrix}^{\boldsymbol{T}} \in \mathbb{S}^2_{++}, \tag{4.26}$$

with

$$\boldsymbol{m}_{1:t-1} \triangleq \left[\boldsymbol{F}^{\boldsymbol{T}}(1)\; \boldsymbol{G}^{\boldsymbol{T}}(1)\; \ldots\; \boldsymbol{F}^{\boldsymbol{T}}(t-1)\; \boldsymbol{G}^{\boldsymbol{T}}(t-1)\right]^{\boldsymbol{T}} \in \mathbb{R}^{2R(t-1)\times 1}, \tag{4.27}$$
$$\boldsymbol{\mu}_{1:t-1} \triangleq [\boldsymbol{\alpha}_S(\mathbf{p}(1))\; \boldsymbol{\alpha}_D(\mathbf{p}(1))\; \ldots\; \boldsymbol{\alpha}_S(\mathbf{p}(t-1))\; \boldsymbol{\alpha}_D(\mathbf{p}(t-1))]^{\boldsymbol{T}}\,\ell \in \mathbb{R}^{2R(t-1)\times 1}, \tag{4.28}$$
$$\boldsymbol{c}^F_{1:t-1}(\mathbf{p}) \triangleq \left[\boldsymbol{c}^F_1(\mathbf{p})\; \ldots\; \boldsymbol{c}^F_{t-1}(\mathbf{p})\right] \in \mathbb{R}^{1\times 2R(t-1)}, \tag{4.29}$$
$$\boldsymbol{c}^G_{1:t-1}(\mathbf{p}) \triangleq \left[\boldsymbol{c}^G_1(\mathbf{p})\; \ldots\; \boldsymbol{c}^G_{t-1}(\mathbf{p})\right] \in \mathbb{R}^{1\times 2R(t-1)}, \tag{4.30}$$
$$\boldsymbol{c}^F_k(\mathbf{p}) \triangleq \left[\left\{\mathbb{E}\left\{\sigma_S(\mathbf{p},t)\,\sigma_S^j(k)\right\}\right\}_{j\in\mathbb{N}^+_R}\; \left\{\mathbb{E}\left\{\sigma_S(\mathbf{p},t)\,\sigma_D^j(k)\right\}\right\}_{j\in\mathbb{N}^+_R}\right], \forall k \in \mathbb{N}^+_{t-1} \tag{4.31}$$
$$\boldsymbol{c}^G_k(\mathbf{p}) \triangleq \left[\left\{\mathbb{E}\left\{\sigma_D(\mathbf{p},t)\,\sigma_S^j(k)\right\}\right\}_{j\in\mathbb{N}^+_R}\; \left\{\mathbb{E}\left\{\sigma_D(\mathbf{p},t)\,\sigma_D^j(k)\right\}\right\}_{j\in\mathbb{N}^+_R}\right], \forall k \in \mathbb{N}^+_{t-1} \textit{ and} \tag{4.32}$$
$$\boldsymbol{\Sigma}_{1:t-1} \triangleq \begin{bmatrix} \boldsymbol{\Sigma}(1,1) & \cdots & \boldsymbol{\Sigma}(1,t-1) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}(t-1,1) & \cdots & \boldsymbol{\Sigma}(t-1,t-1) \end{bmatrix} \in \mathbb{S}^{2R(t-1)}_{++}, \tag{4.33}$$

for all $(\mathbf{p},t) \in \mathcal{S} \times \mathbb{N}^2_{N_T}$*. Further, for any choice of* $(m,n) \in \mathbb{Z} \times \mathbb{Z}$*, the conditional correlation of the fields* $|f(\mathbf{p},t)|^m$ *and* $|g(\mathbf{p},t)|^n$ *relative to* $\mathscr{C}(\mathcal{T}_{t-1})$ *may be expressed in closed form as*

$$\mathbb{E}\left\{|f(\mathbf{p},t)|^m\,|g(\mathbf{p},t)|^n \big| \mathscr{C}(\mathcal{T}_{t-1})\right\}$$

$$\equiv 10^{(m+n)\rho/20}\exp\left(\frac{\log(10)}{20}\begin{bmatrix}m\\n\end{bmatrix}^{\boldsymbol{T}}\boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p})+\left(\frac{\log(10)}{20}\right)^2\begin{bmatrix}m\\n\end{bmatrix}^{\boldsymbol{T}}\boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p})\begin{bmatrix}m\\n\end{bmatrix}\right), \quad (4.34)$$

at any $\mathbf{p}\in\mathcal{S}$ *and for all* $t\in\mathbb{N}^2_{N_T}$.

Proof of Lemma 2. See Appendix C. ∎

Since, by exploitting Lemma 2 and (4.21), formula (4.20) can be evaluated without any particular difficulty, we now propose the replacement of the original pointwise problem of interest, (4.17), with either of the heuristics

$$\begin{array}{ll}\underset{\mathbf{p}}{\text{maximize}} & \dfrac{1}{\mathbb{E}\left\{V_{II}(\mathbf{p},t)|\mathscr{C}(\mathcal{T}_{t-1})\right\}}\\ \text{subject to} & \mathbf{p}\in\mathcal{C}_i(\mathbf{p}^o(t-1))\end{array} \quad (4.35)$$

and

$$\begin{array}{ll}\underset{\mathbf{p}}{\text{maximize}} & \dfrac{\mathbb{E}\left\{\left.(V_{II}(\mathbf{p},t))^2\right|\mathscr{C}(\mathcal{T}_{t-1})\right\}}{\left(\mathbb{E}\left\{V_{II}(\mathbf{p},t)|\mathscr{C}(\mathcal{T}_{t-1})\right\}\right)^3}\ ,\\ \text{subject to} & \mathbf{p}\in\mathcal{C}_i(\mathbf{p}^o(t-1))\end{array} \quad (4.36)$$

to be solved at relay $i\in\mathbb{N}^+_R$, at each time $t-1\in\mathbb{N}^+_{N_T-1}$, depending on the order of approximation employed, respectively. Observe that Jensen's Inequality directly implies that the objective of (4.35) is always *lower than or equal* than that of (4.36) and that of the original program (4.17), conditioned, of course, on identical information. As a result, (4.35) is also a *lower bound relaxation* to (4.17). On the other hand, the objective of (4.35) might be desirable in practice, since it is easier to compute. Both approximations are technically well behaved, though, as made precise by the next theorem.

Theorem 4. (Behavior of Approximation Chains I / SINR Maximization) *Both heuristics (4.35) and (4.36) each feature at least one measurable maximizer. Therefore, provided that any of the two heuristics is solved at each time slot* $t-1\in\mathbb{N}^+_{N_T-1}$*, that the selected one features a unique maximizer,* $\widetilde{\mathbf{p}}^*(t)$*, and that* $\widetilde{\mathbf{p}}^*(t)\equiv\mathbf{p}^o(t)$*, for all* $t\in\mathbb{N}^2_{N_T}$*, the produced decision chain is measurable and condition* **C2** *is satisfied at all times.*

Proof of Theorem 4. See Appendix C. ∎

Theorem 4 implies that, at each time slot $t\in\mathbb{N}^+_{N_T-1}$ and under the respective conditions, the chosen heuristic constitutes a well defined approximation to the original problem, (4.17) and, in turn, to (4.15), in the sense that all conditions **C1-C4** are satisfied.

At this point, it will be important to note that, for *each* $\mathbf{p}\in\mathcal{S}$, computation of the conditional mean and covariance in (4.22) of Lemma 2 require execution of matrix operations, which are of *expanding dimension in* $t\in\mathbb{N}^2_{N_T}$; observe that, for instance, the covariance matrix $\boldsymbol{\Sigma}_{1:t-1}$ is of size $2R(t-1)$, which is increasing in $t\in\mathbb{N}^2_{N_T}$. Fortunately, however, the increase is linear in t. Additionally, the reader may readily observe that the inversion of the covariance matrix $\boldsymbol{\Sigma}_{1:t-1}$ constitutes the computationally dominant operation in the long formulas of Lemma 2. The computational complexity of this matrix inversion, which takes place at each time slot $t-1\in\mathbb{N}^+_{N_T-1}$, is, in general, of the order of $\mathcal{O}\left(R^3t^3\right)$ elementary operations. Fortunately though, we may exploit the

Matrix Inversion Lemma, in order to reduce the computational complexity of the aforementioned matrix inversion to the order of $\mathcal{O}\left(R^3t^2\right)$. Indeed, by construction, $\mathbf{\Sigma}_{1:t-1}$ may be expressed as

$$\mathbf{\Sigma}_{1:t-1} \equiv \begin{bmatrix} \mathbf{\Sigma}_{1:t-2} & \mathbf{\Sigma}^c_{1:t-2} \\ \left(\mathbf{\Sigma}^c_{1:t-2}\right)^{\boldsymbol{T}} & \mathbf{\Sigma}\left(t-1,t-1\right) \end{bmatrix}, \tag{4.37}$$

where

$$\mathbf{\Sigma}^c_{1:t-2} \triangleq \left[\mathbf{\Sigma}\left(1,t-1\right) \, \dots \, \mathbf{\Sigma}\left(t-2,t-1\right)\right]^{\boldsymbol{T}} \in \mathbb{R}^{2R(t-2)\times 2R}. \tag{4.38}$$

Invoking the Matrix Inversion Lemma, we obtain the *recursive* expression

$$\mathbf{\Sigma}^{-1}_{1:t-1} = \begin{bmatrix} \mathbf{\Sigma}^{-1}_{1:t-2} + \mathbf{\Sigma}^{-1}_{1:t-2}\mathbf{\Sigma}^c_{1:t-2}\mathbf{S}^{-1}_{t-1}\left(\mathbf{\Sigma}^c_{1:t-2}\right)^{\boldsymbol{T}}\mathbf{\Sigma}^{-1}_{1:t-2} & -\mathbf{\Sigma}^{-1}_{1:t-2}\mathbf{\Sigma}^c_{1:t-2}\mathbf{S}^{-1}_{t-1} \\ -\mathbf{S}^{-1}_{t-1}\left(\mathbf{\Sigma}^c_{1:t-2}\right)^{\boldsymbol{T}}\mathbf{\Sigma}^{-1}_{1:t-2} & \mathbf{S}^{-1}_{t-1} \end{bmatrix}, \quad \text{with} \tag{4.39}$$

$$\mathbf{S}_{t-1} \triangleq \mathbf{\Sigma}\left(t-1,t-1\right) - \left(\mathbf{\Sigma}^c_{1:t-2}\right)^{\boldsymbol{T}}\mathbf{\Sigma}^{-1}_{1:t-2}\mathbf{\Sigma}^c_{1:t-2} \in \mathbb{S}^{2R}_{++}, \tag{4.40}$$

where $\mathbf{S}_{t-1}$ is the respective Schur complement. From (4.39) and (4.40), it can be easily verified that the most computationally demanding operation involved is $\mathbf{\Sigma}^{-1}_{1:t-2}\mathbf{\Sigma}^c_{1:t-2}$, of order $\mathcal{O}\left(R^3t^2\right)$. Since the inversion of $\mathbf{S}_{t-1}$ is of the order of $\mathcal{O}\left(R^3\right)$, we arrive at a total complexity of $\mathcal{O}\left(R^3t^2\right)$ elementary operations of the recursive scheme presented above, and implemented at each time slot $t-1$. The achieved reduction in complexity is important. In most scenarios, R, the number of relays, will be relatively small and fixed for the whole operation of the system, whereas t, the time slot index, might generally take large values, since it is common for the operational horizon of the system, N_T, to be large. Additionally, the reader may readily observe that the aforementioned covariance matrix is independent of the position at which the channel is predicted, $\mathbf{p}$. As a result, its inversion may be performed just once in each time slot, for all evaluations of the mean and covariance of the Gaussian density in (4.22), for all different choices of $\mathbf{p}$ on a fixed grid (say). Consequently, if the total number of such evaluations is $P \in \mathbb{N}^+$, and recalling that the complexity for a matrix-vector multiplication is quadratic in the dimension of the quantities involved, then, *at worst*, the total computational complexity for channel prediction is of the order of $\mathcal{O}\left(PR^2t^2 + R^3t^2\right)$, *at each* $t-1 \in \mathbb{N}^+_{N_T-1}$. This means that a potential actual computational system would have to be able to execute matrix operations with complexity at most of the order of $\mathcal{O}\left(PR^2N_T^2 + R^3N_T^2\right)$, which constitutes the worst case complexity, *over all N_T time slots.* The analysis above characterizes the complexity for solving either of the heuristics (4.35) and (4.36), if the feasible set $\mathcal{C}_i$ is assumed to be finite, for all $i \in \mathbb{N}^+_R$. Of course, if the quantity RN_T is considered a fixed constant, implying that computation of the mean and covariance in (4.22) is considered the result of a black box with fixed (worst) execution time and with input $\mathbf{p}$, then, at each $t-1 \in \mathbb{N}^+_{N_T-1}$, the total computational complexity for channel prediction is of the order of $\mathcal{O}\left(P\right)$ function evaluations, that is, linear in P.

4.3.2 Brute Force

The second approach to the solution of (4.17), considered in this section, is based on the fact that the objective of the aforementioned program can be evaluated rather efficiently, relying on the *multidimensional Gauss-Hermite Quadrature Rule* [40], which constitutes a readily available routine for numerical integration. It is particularly effective for computing expectations of complicated functions of Gaussian random variables [41]. This is indeed the case here, as shown below.

Leveraging Lemma 2 and as it can also be seen in the proof of Theorem 3 (condition **C6**), the objective of (4.17) can be equivalently represented, for all $t \in \mathbb{N}^2_{N_T}$, via a Lebesgue integral as

$$\mathbb{E}\left\{ V_I(\mathbf{p},t) \middle| \mathscr{C}(\mathcal{T}_{t-1}) \right\} = \int_{\mathbb{R}^2} r(\boldsymbol{x}) \mathcal{N}\left(\boldsymbol{x}; \boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p}), \boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p})\right) \mathrm{d}\boldsymbol{x}, \tag{4.41}$$

for any choice of $\mathbf{p} \in \mathcal{S}$, where $\mathcal{N}(\cdot; \boldsymbol{\mu}, \boldsymbol{\Sigma}) : \mathbb{R}^2 \to \mathbb{R}_{++}$ denotes the bivariate Gaussian density, with mean $\boldsymbol{\mu} \in \mathbb{R}^{2\times 1}$ and covariance $\boldsymbol{\Sigma} \in \mathbb{S}^{2\times 2}_{+}$, and the function $r : \mathbb{R}^2 \to \mathbb{R}_{++}$ is defined exploitting the trick (3.11) as

$$r(\boldsymbol{x}) \equiv r(x_1, x_2) \triangleq \frac{P_c P_0 10^{2\rho/10} \left[\exp(x_1 + x_2)\right]^{\frac{\log(10)}{10}}}{P_0 \sigma_D^2 \left[\exp(x_1)\right]^{\frac{\log(10)}{10}} + P_c \sigma^2 \left[\exp(x_2)\right]^{\frac{\log(10)}{10}} + 10^{-\rho/10} \sigma^2 \sigma_D^2}, \tag{4.42}$$

for all $\boldsymbol{x} \equiv (x_1, x_2) \in \mathbb{R}^2$. Exploitting the Lebesgue integral representation (4.41), it can be easily shown that the conditional expectation may be closely approximated by the double summation formula (see Section IV in [41])

$$\mathbb{E}\left\{ V_I(\mathbf{p},t) \middle| \mathscr{C}(\mathcal{T}_{t-1}) \right\} \approx \sum_{l_1 \in \mathbb{N}^+_M} \varpi_{l_1} \sum_{l_2 \in \mathbb{N}^+_M} \varpi_{l_2} r\left(\sqrt{\boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p})} \boldsymbol{q}_{(l_1,l_2)} + \boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p}) \right), \tag{4.43}$$

where $M \in \mathbb{N}^+$ denotes the quadrature resolution, $\boldsymbol{q}_{(l_1,l_2)} \triangleq \left[q_{l_1}\, q_{l_2}\right]^{\boldsymbol{T}} \in \mathbb{R}^{2\times 1}$ denotes the (l_1, l_2)-th quadrature point and $\boldsymbol{\varpi}_{(l_1,l_2)} \triangleq \left[\varpi_{l_1}\, \varpi_{l_2}\right]^{\boldsymbol{T}} \in \mathbb{R}^{2\times 1}$ denotes respective weighting coefficient, for all $(l_1, l_2) \in \mathbb{N}^+_M \times \mathbb{N}^+_M$. Both sets of quadrature points and weighting coefficients are automatically selected *apriori and independently in each dimension*, via the following simple procedure [41, 42]. Let us define a matrix $\boldsymbol{J} \in \mathbb{R}^{M\times M}$, such that

$$\boldsymbol{J}(i,j) \triangleq \begin{cases} \sqrt{\dfrac{\min\{i,j\}}{2}}, & |j-i| \equiv 1 \\ 0, & \text{otherwise} \end{cases}, \quad \forall (i,j) \in \mathbb{N}^+_M \times \mathbb{N}^+_M. \tag{4.44}$$

That is, $\boldsymbol{J}$ constitutes a hollow, tridiagonal, symmetric matrix. Let the sets $\{\lambda_i(\boldsymbol{J}) \in \mathbb{R}\}_{i \in \mathbb{N}^+_M}$ and $\left\{\boldsymbol{v}_i(\boldsymbol{J}) \in \mathbb{R}^{M\times 1}\right\}_{i \in \mathbb{N}^+_M}$ contain the eigenvalues and *normalized* eigenvectors of $\boldsymbol{J}$, respectively. Then, simply, quadrature points and the respective weighting coefficients are selected independently in each dimension $j \in \{1, 2\}$ as

$$q_{l_j} \equiv \sqrt{2} \lambda_{l_j}(\boldsymbol{J}) \quad \text{and} \tag{4.45}$$

$$\varpi_{l_j} \equiv \left(\boldsymbol{v}_{l_j}(\boldsymbol{J})(1)\right)^2, \quad \forall l_j \in \mathbb{N}^+_M. \tag{4.46}$$

In (4.46), $\boldsymbol{v}_{l_j}(\boldsymbol{J})(1)$ denotes the first entry of the involved vector.

Under the above considerations, in this subsection, we propose, for a sufficiently large number of quadrature points M, the replacement of the original pointwise problem (4.17) with the heuristic

$$\begin{array}{ll} \underset{\mathbf{p}}{\text{maximize}} & \displaystyle\sum_{(l_1,l_2) \in \mathbb{N}^+_M \times \mathbb{N}^+_M} \varpi_{l_1} \varpi_{l_2} r\left(\sqrt{\boldsymbol{\Sigma}^{F,G}_{t|t-1}(\mathbf{p})} \boldsymbol{q}_{(l_1,l_2)} + \boldsymbol{\mu}^{F,G}_{t|t-1}(\mathbf{p}) \right) \\ \text{subject to} & \mathbf{p} \in \mathcal{C}_i(\mathbf{p}^o(t-1)) \end{array}, \tag{4.47}$$

to be solved at relay $i \in \mathbb{N}_R^+$, at each time $t-1 \in \mathbb{N}_{N_T-1}^+$. As in Section 4.3.1 above, the following result is in power, concerning the technical consistency of the decision chain produced by considering the approximate program (4.47), for all $t \in \mathbb{N}_{N_T}^2$. Proof is omitted, as it is essentially identical to that of Theorem 4.

Theorem 5. (Behavior of Approximation Chains II / SINR Maximization) *Consider the the heuristic (4.47). Then, under the same circumstances, all conclusions of Theorem 4 hold true.*

Since the computations in (4.45) and (4.46) do not depend on $\mathbf{p}$ or the information collected so far, encoded in $\mathscr{C}(\mathcal{T}_{t-1})$, for $t \in \mathbb{N}_{N_T}^2$, quadrature points and the respective weights can be determined offline and stored in memory. Therefore, the computational burden of (4.43) concentrates solely on the computation of an inner product, whose computational complexity is of the order of $\mathcal{O}\left(M^2\right)$, as well as a total of M^2 evaluations of $r\left(\sqrt{\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})}\boldsymbol{q}_{(l_1,l_2)}+\boldsymbol{\mu}_{t|t-1}^{F,G}(\mathbf{p})\right)$, for *each* value of $\mathbf{p}$. Excluding temporarily the computational burden of $\boldsymbol{\mu}_{t|t-1}^{F,G}(\mathbf{p})$ and $\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})$, each of the latter evaluations is of fixed complexity, since each involves elementary operations among matrices and vectors in $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{2\times 1}$, respectively and, additionally, the involved matrix square root can be evaluated in closed form, via the formula [43]

$$\sqrt{\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})} \equiv \frac{\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})+\sqrt{\det\left(\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})\right)}\mathbf{I}_2}{\sqrt{\operatorname{tr}\left(\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})\right)+2\sqrt{\det\left(\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})\right)}}} \in \mathbb{S}_+^{2\times 2}, \tag{4.48}$$

where we have taken into account that $\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})$ is always a (conditional) covariance matrix and, thus, (conditionally) positive semidefinite. As a result and considering the last paragraph of Section 4.3.1, if (4.43) is evaluated on a finite grid of possible locations, say $P \in \mathbb{N}^+$, then, at each $t-1 \in \mathbb{N}_{N_T-1}^+$, the total computational complexity of the Gauss-Hermite Quadrature Rule outlined above is of the order of $\mathcal{O}\left(PM^2+PR^2t^2+R^3t^2\right)$ elementary operations / function evaluations. This will be the total, worst case computational complexity for solving (4.47), if the feasible set $\mathcal{C}_i$ is assumed to be finite, for all $i \in \mathbb{N}_R^+$. As noted above, a finite feasible set greatly simplifies implementation, since a trial-and-error approach may be employed for solving the respective optimization problem. If M is considered a fixed constant (e.g., $M \equiv 10^3$), and the same holds for $Rt \leq RN_T$, then, in each time slot, the total complexity of the Gauss-Hermite Quadrature Rule is of the order of $\mathcal{O}(P)$ evaluations of (4.43), that is, linear in P. In that case, the whole numerical integration routine is considered a black box of fixed computational load, which, in each time slot, takes $\mathbf{p}$ as its input. Observe that, whenever $M \approx RN_T$, the worst case complexity of the brute force method, described in this subsection, *over all N_T time slots*, is essentially the same as that of the Taylor approximation method, presented earlier in Section 4.3.1.

4.4 Theoretical Guarantees: Network QoS Increases Across Time Slots

The proposed relay position selection approach presented in Section 4.3 enjoys a very important and useful feature, initially observed via numerical simulations: Although a 2-stage stochastic programming procedure is utilized *independently* at each time slot for determining optimal relay positioning and beamforming weights at the next time slot, *the average network QoS (that is, the achieved*

SINR) actually increases, as a function of time (the time slot). Then, it was somewhat surprising to discover that, additionally, this behavior of the achieved SINR can be predicted *theoretically*, in an indeed elegant manner and, as it will be clear below, under mild and reasonable assumptions on the structure of the spatially controlled beamforming problem under consideration. But first, it would be necessary to introduce the following definition.

Definition 2. (L.MD.G Fields) On $(\Omega, \mathscr{F}, \mathcal{P})$, an integrable stochastic field $\Xi : \Omega \times \mathbb{R}^N \times \mathbb{N} \to \mathbb{R}$ is said to be a *Linear Martingale Difference (MD) Generator, relative to a filtration* $\{\mathscr{H}_t \subseteq \mathscr{F}\}_{t\in\mathbb{N}}$, and *with scaling factor* $\mu \in \mathbb{R}$, or, equivalently, $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{H}_t, \mu)$, if and only if, for each $t \in \mathbb{N}^+$, there exists a measurable set $\Omega_t \subseteq \Omega$, with $\mathcal{P}(\Omega_t) \equiv 1$, such that, for every $\boldsymbol{x} \in \mathbb{R}^N$, it is true that

$$\mathbb{E}\left\{\Xi(\boldsymbol{x}, t) | \mathscr{H}_{t-1}\right\}(\omega) \equiv \mu \mathbb{E}\left\{\Xi(\boldsymbol{x}, t-1) | \mathscr{H}_{t-1}\right\}(\omega), \tag{4.49}$$

for all $\omega \in \Omega_t$.

Remark 5. A fine detail in the definition of a $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{H}_t, \mu)$ field is that, for each $t \in \mathbb{N}$, the event Ω_t *does not depend* on the choice of point $\boldsymbol{x} \in \mathbb{R}^N$. Nevertheless, even if the event where (4.49) is satisfied is indeed dependent on the particular $\boldsymbol{x} \in \mathbb{R}^N$, let us denote it as $\Omega_{\boldsymbol{x},t}$, we may leverage the fact that conditional expectations are unique almost everywhere, and *arbitrarily define*

$$\mathbb{E}\left\{\Xi(\boldsymbol{x}, t) | \mathscr{H}_{t-1}\right\}(\omega) \triangleq \mu \mathbb{E}\left\{\Xi(\boldsymbol{x}, t-1) | \mathscr{H}_{t-1}\right\}(\omega), \tag{4.50}$$

for all $\omega \in \Omega^c_{\boldsymbol{x},t}$, where $\mathcal{P}\left(\Omega^c_{\boldsymbol{x},t}\right) \equiv 0$. That is, we modify *both, or either of* the random elements $\mathbb{E}\left\{\Xi(\boldsymbol{x}, t-1) | \mathscr{H}_{t-1}\right\}$ and $\mathbb{E}\left\{\Xi(\boldsymbol{x}, t) | \mathscr{H}_{t-1}\right\}$, on the null set $\Omega^c_{\boldsymbol{x},t}$, such that (4.49) is satisfied. Then, it may be easily verified that both such modifications result in valid versions of the conditional expectations of $\Xi(\boldsymbol{x}, t-1)$ and $\Xi(\boldsymbol{x}, t)$ relative to $\mathscr{H}_{t-1}$, respectively and satisfy property (4.49), *everywhere with respect to* $\omega \in \Omega$.

In Definition 2, invariance of Ω_t with respect to $\boldsymbol{x} \in \mathbb{R}^N$, in conjunction with the power of the substitution rule for conditional expectations (Section 8.2.1), will allow the development of strong conditional arguments, *when $\boldsymbol{x}$ is replaced by a random element, measurable relative to $\mathscr{H}_{t-1}$.* ■

Remark 6. There are lots of examples of **L.MD.G** stochastic fields, satisfying the technical properties of Definition 2. For completeness, let us present two such examples. Employing generic notation, consider an integrable real-valued stochastic field $Y(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^N$, $t \in \mathbb{N}$. Let the natural filtration associated with $Y(\boldsymbol{x}, t)$ be $\{\mathscr{Y}_t\}_{t\in\mathbb{N}}$, with $\mathscr{Y}_t \triangleq \sigma\left\{Y(\boldsymbol{x}, t), \boldsymbol{x} \in \mathbb{R}^N\right\}$, for all $t \in \mathbb{N}$. Also, consider another, for simplicity temporal, integrable real-valued process $W(t), t \in \mathbb{N}$. Suppose, further, that $Y(\boldsymbol{x}, t)$ is a martingale with respect to $t \in \mathbb{N}$ (relative to $\{\mathscr{Y}_t\}_{t\in\mathbb{N}}$), and that $W(t)$ is a zero mean process, independent of $Y(\boldsymbol{x}, t)$. In particular, we assume that, for every $t \in \mathbb{N}^+$, there exist events $\Omega^Y_t \subseteq \Omega$ and $\Omega^W_t \subseteq \Omega$, satisfying $\mathcal{P}\left(\Omega^Y_t\right) \equiv 1$ and $\mathcal{P}\left(\Omega^W_t\right) \equiv 1$, such that, for all $\boldsymbol{x} \in \mathbb{R}^N$,

$$\mathbb{E}\left\{Y(\boldsymbol{x}, t) | \mathscr{Y}_{t-1}\right\}(\omega) \equiv Y(\omega, \boldsymbol{x}, t-1) \quad \text{and} \tag{4.51}$$

$$\mathbb{E}\left\{W(t) | \mathscr{Y}_{t-1}\right\}(\omega) \equiv 0, \tag{4.52}$$

for all $\omega \in \Omega^Y_t \bigcap \Omega^W_t$, where, apparently, $\mathcal{P}\left(\Omega^Y_t \bigcap \Omega^W_t\right) \equiv 1$.

Our first, probably most basic example of a **L.MD.G** field is simply the martingale $Y(\boldsymbol{x}, t)$ itself. Of course, in order to verify this statement, we need to show that it satisfies the technical requirements of Definition 2, relative to a given filtration; in particular, let us choose $\{\mathscr{Y}_t\}_{t\in\mathbb{N}}$ to be that filtration. Then, for every $(\boldsymbol{x},t)\in\mathbb{R}^N\times\mathbb{N}^+$, it is trivial to see that

$$\mathbb{E}\left\{ Y(\boldsymbol{x},t)|\,\mathscr{Y}_{t-1}\right\}(\omega)\equiv Y(\omega,\boldsymbol{x},t-1)\equiv\mathbb{E}\left\{ Y(\boldsymbol{x},t-1)|\,\mathscr{Y}_{t-1}\right\}(\omega), \tag{4.53}$$

for all $\omega\in\Omega_t^Y$, where $Y(\boldsymbol{x},t-1)$ *is chosen as our version of* $\mathbb{E}\{Y(\boldsymbol{x},t-1)|\,\mathscr{Y}_{t-1}\}$, *everywhere* in Ω. As a result, the martingale $Y(\boldsymbol{x},t)$ is itself a $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{Y}_t,1)$, as expected.

The second, somewhat more interesting example of a **L.MD.G** field is defined as

$$X(\boldsymbol{x},t)\triangleq\varrho Y(\boldsymbol{x},t)+W(t), \tag{4.54}$$

for all $(\boldsymbol{x},t)\in\mathbb{R}^N\times\mathbb{N}$, where, say, $0<\varrho\leq 1$. In order to verify the technical requirements of Definition 2, let us again choose $\{\mathscr{Y}_t\}_{t\in\mathbb{N}}$ as our filtration. Then, for every $(\boldsymbol{x},t)\in\mathbb{R}^N\times\mathbb{N}^+$, there exists a measurable set $\Omega_{\boldsymbol{x},t}^{Y,W}\subseteq\Omega$, with $\mathcal{P}\left(\Omega_{\boldsymbol{x},t}^{Y,W}\right)\equiv 1$, such that, for all $\omega\in\Omega_{\boldsymbol{x},t}^{Y,W}$,

$$\begin{aligned}\mathbb{E}\left\{ X(\boldsymbol{x},t)|\,\mathscr{Y}_{t-1}\right\}(\omega)&\equiv\varrho Y(\omega,\boldsymbol{x},t-1)+\mathbb{E}\{W(t)\}\\&\equiv\varrho Y(\omega,\boldsymbol{x},t-1).\end{aligned} \tag{4.55}$$

Therefore, we may choose our version for $\mathbb{E}\left\{ X(\boldsymbol{x},t)|\,\mathscr{Y}_{t-1}\right\}$ as

$$\mathbb{E}\left\{ X(\boldsymbol{x},t)|\,\mathscr{Y}_{t-1}\right\}(\omega)\equiv\varrho Y(\omega,\boldsymbol{x},t-1),\quad\forall\omega\in\Omega. \tag{4.56}$$

In exactly the same fashion, we may choose, for every $(\boldsymbol{x},t)\in\mathbb{R}^N\times\mathbb{N}^+$,

$$\mathbb{E}\left\{ X(\boldsymbol{x},t-1)|\,\mathscr{Y}_{t-1}\right\}(\omega)\equiv\varrho Y(\omega,\boldsymbol{x},t-1),\quad\forall\omega\in\Omega. \tag{4.57}$$

Consequently, for every $(\boldsymbol{x},t)\in\mathbb{R}^N\times\mathbb{N}^+$, it will be true that

$$\mathbb{E}\left\{ X(\boldsymbol{x},t)|\,\mathscr{Y}_{t-1}\right\}(\omega)\equiv\varrho Y(\omega,\boldsymbol{x},t-1)\equiv\mathbb{E}\left\{ X(\boldsymbol{x},t-1)|\,\mathscr{Y}_{t-1}\right\}(\omega), \tag{4.58}$$

for all $\omega\in\Omega$, showing that the field $X(\boldsymbol{x},t)$ is also $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{Y}_t,1)$. ■

Leveraging the notion of a **L.MD.G** field, the following result may be proven, characterizing the temporal (in discrete time) evolution of the objective of myopic stochastic programs of the form of (4.5). In order to introduce the result, let us consider the family $\left\{\mathscr{P}_t^{\uparrow}\right\}_{t\in\mathbb{N}_{N_T}^+}$, with $\mathscr{P}_t^{\uparrow}$ being the *limit σ-algebra generated by all admissible policies at time slot t,* defined as

$$\mathscr{P}_t^{\uparrow}\triangleq\sigma\left\{\bigcup_{\mathbf{p}(t)\in\mathcal{D}_t}\sigma\{\mathbf{p}(t)\}\right\}\subseteq\mathscr{C}(\mathcal{T}_{t-1}),\quad\forall t\in\mathbb{N}_{N_T}^+, \tag{4.59}$$

with $\mathscr{P}_1^{\uparrow}$ being the trivial σ-algebra; recall that $\mathbf{p}(1)\in\mathcal{S}^R$ is assumed to be a constant. Also, for every $t\in\mathbb{N}_{N_T}^+$, let us define the class

$$\overline{\mathcal{D}}_t\equiv\left\{\mathbf{p}:\Omega\to\mathcal{S}^R\,\middle|\,\mathbf{p}^{-1}(\mathcal{A})\in\mathscr{P}_t^{\uparrow},\text{ for all }\mathcal{A}\in\mathscr{B}\left(\mathcal{S}^R\right)\right\}. \tag{4.60}$$

The result now follows.

Theorem 6. (L.MD.G Objectives Increase over Time) *Consider, for each $t \in \mathbb{N}^2_{N_T}$, the maximization version of the 2-stage stochastic program (4.5), for some choice of the second-stage optimal value $V(\mathbf{p}, t)$, $\mathbf{p} \in \mathcal{S}^R$, $t \in \mathbb{N}^2_{N_T}$. Suppose that conditions* **C1-C6** *are satisfied at all times and let $\mathbf{p}^*(t)$ denote an optimal solution to (4.5), decided at $t-1 \in \mathbb{N}^+_{N_T-1}$. Suppose, further, that, for every $t \in \mathbb{N}^+_{N_T}$,*

- *$V(\mathbf{p}, t)$ is $\mathbf{L.MD.G}\Diamond(\mathscr{H}_t, \mu)$, for a filtration $\left\{\mathscr{H}_t \supseteq \mathscr{P}^{\uparrow}_t\right\}_{t \in \mathbb{N}^+_{N_T}}$ and some $\mu \in \mathbb{R}$, and that*
- *$V(\cdot, \cdot, t)$ is both $\boldsymbol{SP}\Diamond\mathfrak{C}_{\mathscr{H}_t}$ and $\boldsymbol{SP}\Diamond\mathfrak{C}_{\mathscr{H}_{t-1}}$, with $\overline{\mathcal{D}}_t \subseteq \mathfrak{C}_{\mathscr{H}_t} \subseteq \mathfrak{I}_{\mathscr{H}_t}$ (Remark 11 / Section 8.2.1).*

Then, for any admissible policy $\mathbf{p}^o(t-1)$, it is true that

$$\mu\mathbb{E}\left\{V\left(\mathbf{p}^o(t-1), t-1\right)\right\} \equiv \mathbb{E}\left\{V\left(\mathbf{p}^o(t-1), t\right)\right\} \quad \text{and} \tag{4.61}$$

$$\mu\mathbb{E}\left\{V\left(\mathbf{p}^*(t-1), t-1\right)\right\} \leq \mathbb{E}\left\{V\left(\mathbf{p}^*(t), t\right)\right\}, \quad \forall t \in \mathbb{N}^2_{N_T}. \tag{4.62}$$

In particular, if $\mu \equiv 1$, the objective: • does not decrease by not updating the decision variable, and • is nondecreasing over time, under optimal decision making.

Proof of Theorem 6. See Appendix C. ■

Of all possible choices for μ, the one where $\mu \equiv 1$ is of special importance and practical relevance, as we will see in the next section. In particular, in this case, and provided that the respective assumptions are fulfilled, Theorem 6 implies that *optimal myopic exploration of the random field $V(\mathbf{p}, t)$ is monotonic*, either under optimal decision making, or by retaining the same policy next.

In the case where $\mu \neq 1$, things can be quite interesting as well. For instance, suppose that one focuses on the maximization counterpart of the stochastic program (4.5). In this case, it is of interest to sequentially, myopically and feasibly sample the field $V(\mathbf{p}, t)$, such that it is *maximized on average*. Let us also refer to $V(\mathbf{p}, t)$ as the *reward* of the sampling process. Additionally, suppose that $V(\mathbf{p}, t)$ is a **L.MD.G** field, with parameter $\mu \equiv 0.9 < 1$. Assuming that the respective assumptions are satisfied, Theorem 6 implies that, for any admissible sampling policy $\mathbf{p}^o(t-1)$,

$$\mathbb{E}\{V\left(\mathbf{p}^o(t-1), t\right)\} \equiv 0.9\mathbb{E}\{V\left(\mathbf{p}^o(t-1), t-1\right)\} \text{ and} \tag{4.63}$$

$$\mathbb{E}\left\{V\left(\mathbf{p}^*(t), t\right)\right\} \geq 0.9\mathbb{E}\left\{V\left(\mathbf{p}^*(t-1), t-1\right)\right\}, \tag{4.64}$$

for all $t \in \mathbb{N}^2_{N_T}$. In other words, either performing optimal decision making, or retaining the same policy next, will result in an *at most* 10% *loss of performance*. This means that the performance of optimal sampling at the next time step cannot be worse than 90% of that at the current time slot. Of course, this is important, because, in a sense, *the risk of (non-)maintaining the average reward achieved up to the current time slot is meaningfully quantified.*

Remark 7. When the stochastic program under study is *separable*, that is, when the objective is of the form

$$V(\mathbf{p}(t), t) \equiv \sum_{i \in \mathbb{N}^+_M} V_i(\mathbf{p}_i(t), t) \tag{4.65}$$

(and the respective constraints of the problem decoupled), then, in order to reach the conclusions of Theorem 6 for V, it *suffices* for Theorem 6 to hold *individually* for each V_i, $i \in \mathbb{N}^+_M$. This is true, for instance, for the spatially controlled beamforming problem (4.15). ■

We may now return to the beamforming problem under consideration, namely (4.15). By Remark 7 and Theorem 6, it would suffice if we could show that the field $V_I(\mathbf{p},t)$ is a linear MD generator, relative to a properly chosen filtration. Unfortunately, though, this does not seem to be the case; the statistical structure of $V_I(\mathbf{p},t)$ does not match that of a linear MD generator *exactly*, relative to any reasonably chosen filtration. Nevertheless, under the channel model of Section 3, it is indeed possible to show that $V_I(\mathbf{p},t)$ is *approximately* $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{C}(\mathcal{T}_{t-1}),1)$, a fact that explains, in an elegant manner, why our proposed spatially controlled beamforming framework is expected to work so well, both under optimal and suboptimal decision making.

To show that $V_I(\mathbf{p},t)$ is *approximately* $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{C}(\mathcal{T}_{t-1}),1)$, simply consider projecting $V_I(\mathbf{p},t-1)$ onto $\mathscr{C}(\mathcal{T}_{t-2})$, via the conditional expectation $\mathbb{E}\{V_I(\mathbf{p},t-1)|\mathscr{C}(\mathcal{T}_{t-2})\}$. Of course, and based on what we have seen so far, $\mathbb{E}\{V_I(\mathbf{p},t-1)|\mathscr{C}(\mathcal{T}_{t-2})\}$ can be written as a Lebesgue integral of $V_I(\mathbf{p},t-1)$ expressed in terms of the vector field $[F(\mathbf{p},t-1)\ G(\mathbf{p},t-1)]^{\boldsymbol{T}}$, times its conditional density relative to $\mathscr{C}(\mathcal{T}_{t-2})$. It then easy to see that this conditional density will be, of course, Gaussian, and will be of exactly the same form as the conditional density of $[F(\mathbf{p},t)\ G(\mathbf{p},t)]^{\boldsymbol{T}}$ relative to $\mathscr{C}(\mathcal{T}_{t-1})$, as presented in Lemma 2, but with t replaced by $t-1$. Likewise, $\mathbb{E}\{V_I(\mathbf{p},t)|\mathscr{C}(\mathcal{T}_{t-2})\}$ is of the same form as $\mathbb{E}\{V_I(\mathbf{p},t-1)|\mathscr{C}(\mathcal{T}_{t-2})\}$, but with all terms

$$\exp\left(-\frac{1}{\gamma}\right),\exp\left(-\frac{2}{\gamma}\right),\ldots,\exp\left(-\frac{t-2}{\gamma}\right) \tag{4.66}$$

simply replaced by

$$\exp\left(-\frac{2}{\gamma}\right),\exp\left(-\frac{3}{\gamma}\right),\ldots,\exp\left(-\frac{t-1}{\gamma}\right), \tag{4.67}$$

for all $t\in\mathbb{N}^3_{N_T}$. Of course, if $t\equiv 2$, we have

$$\begin{aligned}\mathbb{E}\{V_I(\mathbf{p},2)|\mathscr{C}(\mathcal{T}_0)\} &\equiv \mathbb{E}\{V_I(\mathbf{p},2)\}\\ &\equiv \mathbb{E}\{V_I(\mathbf{p},1)\}\equiv \mathbb{E}\{V_I(\mathbf{p},1)|\mathscr{C}(\mathcal{T}_0)\}.\end{aligned} \tag{4.68}$$

Now, for γ *sufficiently large*, we may approximately write

$$\exp\left(-\frac{x+1}{\gamma}\right)\approx\exp\left(-\frac{x}{\gamma}\right),\quad \forall x>1, \tag{4.69}$$

and, therefore, due to continuity, it should be true that

$$\mathbb{E}\{V_I(\mathbf{p},t)|\mathscr{C}(\mathcal{T}_{t-2})\}\approx\mathbb{E}\{V_I(\mathbf{p},t-1)|\mathscr{C}(\mathcal{T}_{t-2})\}, \tag{4.70}$$

for all $t\in\mathbb{N}^2_{N_T}$ (and everywhere with respect to $\omega\in\Omega$). As a result, we have shown that, *at least approximately*, $V_I(\mathbf{p},t)$ is $\mathbf{L}.\mathbf{MD}.\mathbf{G}\diamondsuit(\mathscr{C}(\mathcal{T}_{t-1}),1)$. We may then invoke Theorem 6 in an approximate manner, leading to the following important result. Hereafter, for $x\in\mathbb{R}$ and $y\in\mathbb{R}$, $x\lesssim y$ will imply that x *is approximately smaller or equal than* y, in the sense that $x\le y+\varepsilon$, where $\varepsilon>0$ is some small slack.

Theorem 7. **(QoS Increases over Time Slots)** *Consider the separable stochastic program (4.15). For γ sufficiently large, and for any admissible policy $\mathbf{p}^o(t-1)$, it is true that*

$$\mathbb{E}\{V_I(\mathbf{p}_i^o(t-1),t-1)\}\approx\mathbb{E}\{V_I(\mathbf{p}_i^o(t-1),t)\}, \tag{4.71}$$

$$\mathbb{E}\left\{V_I\big(\mathbf{p}_i^*(t-1),t-1\big)\right\}\lesssim\mathbb{E}\left\{V_I\big(\mathbf{p}_i^*(t),t\big)\right\},\quad \forall i\in\mathbb{N}_R^+ \tag{4.72}$$

$$\mathbb{E}\left\{V\left(\mathbf{p}^{o}\left(t-1\right),t-1\right)\right\} \approx \mathbb{E}\left\{V\left(\mathbf{p}^{o}\left(t-1\right),t\right)\right\} \quad \textit{and} \tag{4.73}$$

$$\mathbb{E}\left\{V\big(\mathbf{p}^{*}\left(t-1\right),t-1\big)\right\} \lesssim \mathbb{E}\left\{V\big(\mathbf{p}^{*}\left(t\right),t\big)\right\}, \tag{4.74}$$

for all $t \in \mathbb{N}^2_{N_T}$. In other words, ***approximately****, the average network QoS: • does not decrease by not updating the positions of the relays and • is nondecreasing across time slots, under (per relay) optimal decision making.*

Theorem 7 is very important from a practical point of view, and has the following additional implications. Roughly speaking, under the conditions of Theorem 7, that is, if the temporal interactions of the channel are sufficiently strong, the average network QoS is not (approximately) expected to, at least *abruptly,* decrease if one or more relays stop moving at some point. Such event might indeed happen in an actual autonomous network, possibly due to power limitations, or a failure in the motion mechanisms of some network nodes. In the same framework, Theorem 7 implies that the relays which continue moving contribute (approximately) positively to increasing the average network QoS, across time slots. Such behavior of the proposed spatially controlled beamforming system may be also confirmed numerically, as discussed in Section 5. For the record, and as it will be also shown in Section 5, relatively small values for the correlation time γ, such as $\gamma \equiv 5$, are sufficient in order to practically observe the nice system behavior promised by Theorem 7. This fact makes the proposed spatially controlled beamforming system attractive in terms of practical feasibility, and shows that such an approach could actually enhance system performance in a well-behaved, real world situation.

5 Numerical Simulations & Experimental Validation

In this section, we present synthetic numerical simulations, which essentially confirm that the proposed approach, previously presented in Section 4, actually works, and results in relay motion control policies, which yield improved beamforming performance. All synthetic experiments were conducted on an imaginary square terrain of dimensions 30×30 squared units of length, with $\mathcal{W} \equiv [0, 30]^2$, uniformly divided into $30 \times 30 \equiv 900$ square regions. The locations of the source and destination are fixed as $\mathbf{p}_S \equiv [15\,0]^{\boldsymbol{T}}$ and $\mathbf{p}_D \equiv [15\,30]^{\boldsymbol{T}}$. The beamforming temporal horizon is chosen as $T \equiv 40$ and the number of relays is fixed at $R \equiv 8$. The wavelength is chosen as $\lambda \equiv 0.125$, corresponding to a carrier frequency of $2.4\,GHz$. The various parameters of the assumed channel model are set as $\ell \equiv 3$, $\rho \equiv 20$, $\sigma_\xi^2 \equiv 20$, $\eta^2 \equiv 50$, $\beta \equiv 10$, $\gamma \equiv 5$ and $\delta \equiv 1$. The variances of the reception noises at the relays and the destination are fixed as $\sigma^2 \equiv \sigma_D^2 \equiv 1$. Lastly, both the transmission power of the source and the *total* transmission power budget of the relays are chosen as $P \equiv P_c \equiv 25$ ($\approx 14dB$) units of power.

The relays are restricted to the rectangular region $\mathcal{S} \equiv [0, 30] \times [12, 18]$. Further, at each time instant, each of the relays is allowed to move inside a 9-region area, centered at each current position, thus defining its closed set of feasible directions $\mathcal{C}_i$, for each relay $i \in \mathbb{N}^+_R$. Basic collision and out-of-bounds control was also considered and implemented.

In order to assess the effectiveness of our proposed approach, we compare both heuristics (4.35) and (4.36) against the case where an *agnostic*, *purely randomized* relay control policy is adopted; in this case, at each time slot, each relay moves randomly to a new available position, without taking previously observed CSI into consideration. For simplicity, we do not consider the brute force method presented earlier in Section 4.3.2. For reference, we also consider the performance of an oracle control policy at the relays, where, at each time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, relay $i \in \mathbb{N}^+_R$

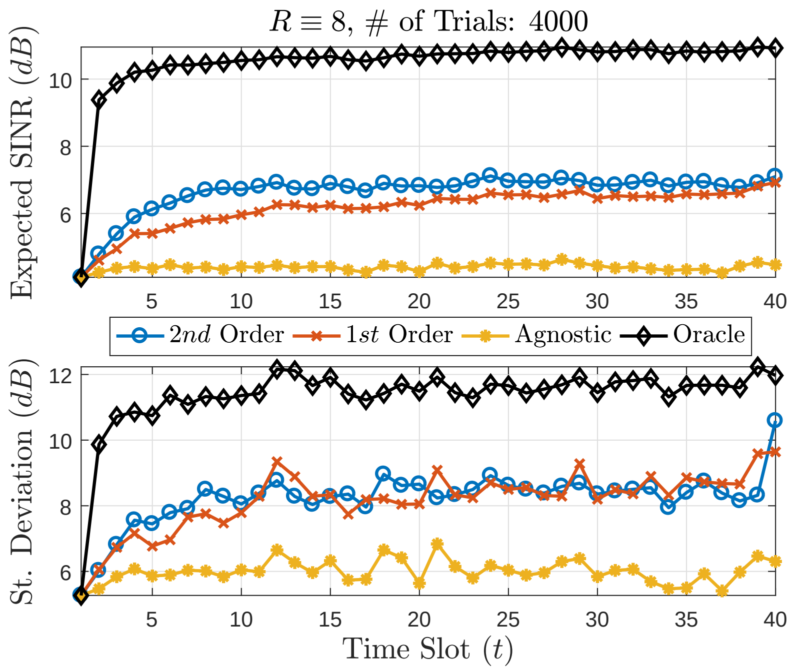

Figure 5.1: Comparison of the proposed strategic relay planning schemes, versus an agnostic, randomized relay motion policy.

updates its position by *noncausally* looking into the future and choosing the position $\mathbf{p}_i$, which maximizes directly the quantity $V_I(\mathbf{p}_i, t)$, over $\mathcal{C}_i(\mathbf{p}_i(t-1))$. Of course, the comparison of all controlled systems is made under exactly the same communication environment.

Fig. 5.1 shows the expectation and standard deviation of the achieved QoS for all controlled systems, approximated by executing 4000 trials of the whole experiment. As seen in the figure, there is a clear advantage in exploiting strategically designed relay motion control. Whereas the agnostic system maintains an average SINR of about $4\,dB$ at all times, the system based on the proposed $2nd$ order heuristic (4.36) is clearly superior, exhibiting an increasing trend in the achieved SINR, with a gap starting from about $0.5\,dB$ at time slot $t \equiv 2$, up to $3\,dB$ at time slots $t \equiv 10, 11, \ldots, 40$. The $1st$ order heuristic (4.35) comes second, with always slightly lower average SINR, and which also exhibits a similar increasing trend as the $2nd$ order heuristic (4.36). Additionally, it seems to converge to the performance achieved by (4.36), across time slots. The existence of an increasing trend in the achieved average network QoS has already been predicted by Theorem 6 *for a strictly optimal policy*, and our experiments confirm this behavior for both heuristics (4.35) and (4.36), as well. This shows that both heuristics constitute excellent approximations to the original problem (4.17). Consequently, it is both theoretically and experimentally verified that, although the proposed stochastic programming formulation is essentially myopic, the resulting system performance is not, and this is dependent on the fact that the channel exhibits non trivial temporal statistical interactions. We should also comment on the standard deviation of all systems, which, from Fig. 5.1, seems somewhat high, relative to the range of the respective average SINR. This is *exclusively* due to the wild variations of the channel, which, in turn, are due to the effects of shadowing and multipath fading; it is *not* due to the adopted beamforming technique. This is reasonable, since, when the channel is not *actually* in deep fade at time t (an event which might happen with positive

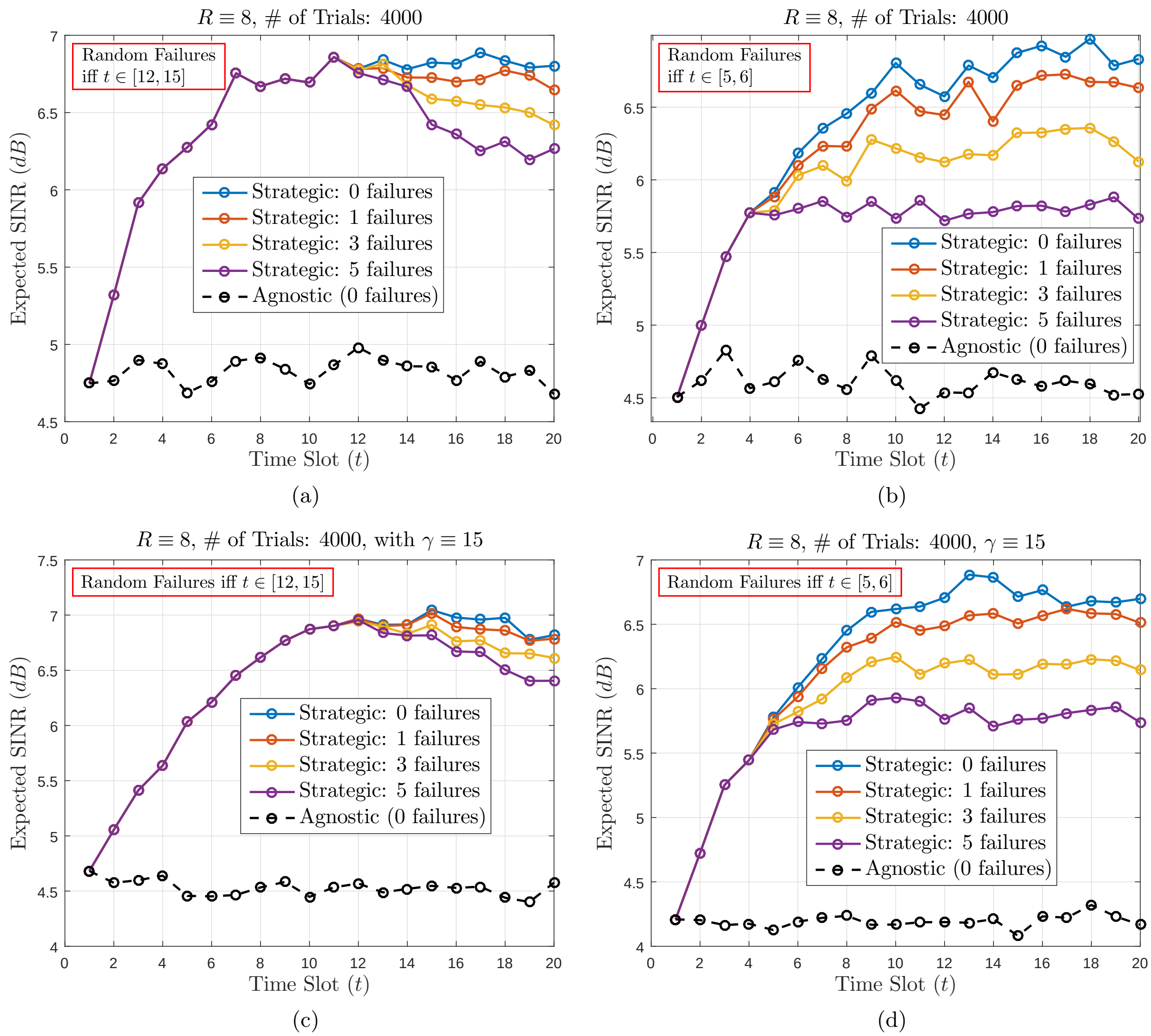

Figure 5.2: Performance of the proposed spatially controlled system, at the presence of motion failures.

probability), the relays, at time $t-1$, are predictively steered to locations, which, most probably, incur higher network QoS. As clearly shown in Fig. 5.1, for all systems under study, including that implementing the oracle policy, an increase in system performance also implies a proportional increase in the respective standard deviation.

Next, we experimentally evaluate the performance of the system at the presence of *random motion failures* in the network. Hereafter, we work with the $2nd$ order heuristic (4.36), and set $T \equiv 20$. Random motion failures are modeled by choosing, at each trial, a *random sample* of a fixed number of relays and a *random time* when the failures occur, that is, at each time, the selected relays just stop moving; they continue to beamform staying still, at the position each of them visited last. Two cases are considered; in the first case, motion failures happen if and only if $t \in [12, 15]$ (Figs. 5.2a and 5.2c), whereas, in the second case, $t \in [5, 6]$ (Figs. 5.2b and 5.2d). In both cases, zero, one, three and five relays (chosen at random, at each trial) stop moving. Two cases for γ are considered, $\gamma \equiv 5$ (Figs. 5.2a and 5.2b) and $\gamma \equiv 15$ (Figs. 5.2c and 5.2d).

Again, the results presented in Fig. 5.2 pleasingly confirm our predictions implied by Theorem 6 (note, however, that Theorem 6 does not support *randomized* motion failures; on the other hand, our simulations are such in order to stress test the proposed system in more adverse motion failure cases). In particular, Fig. 5.2a clearly demonstrates that a larger number of motion failures induces a proportional, relatively (depending on γ) slight decrease in performance; this decrease, though, is smoothly evolving, and is not abrupt. This behavior is more pronounced in Fig. 5.2c, where the correlation time parameter γ has been increased to 15 (recall that, in Theorem 6, γ is assumed to be sufficiently large). We readily observe that, in this case, over the same horizon, the operation of the system is smoother, and decrease in performance, as well as its slope, are significantly smaller than those in Fig. 5.2a, for all cases of motion failures. Now, in Figs. 5.2b and 5.2d, when motion failures happen early, well before the network QoS converges to its maximal value, we observe that, although some relays might stop moving at some point, the achieved expected network QoS continues exhibiting its usual increasing trend. Of course, the performance of the system converges values strictly proportional to the number of failures in each of the cases considered. This means that the relays which continue moving contribute positively to increasing network QoS. This has been indeed predicted by Theorem 6, as well.

6 Conclusions

We have considered the problem of enhancing QoS in time slotted relay beamforming networks with one source/destination, via stochastic relay motion control. Modeling the wireless channel as a spatiotemporal stochastic field, we proposed a novel 2-stage stochastic programming formulation for predictively specifying relay positions, such that the future expected network QoS is maximized, based on causal CSI and under a total relay power constraint. We have shown that this problem can be effectively approximated by a set of simple, two dimensional subproblems, which can be distributively solved, one at each relay. System optimality was tediously analyzed under a rigorous mathematical framework, and our analysis resulted in the development of an extended version of the Fundamental Lemma of Stochastic Control, which constitutes a result of independent interest, as well. We have additionally provided strong theoretical guarantees, characterizing the performance of the proposed system, and showing that the average QoS achieved improves over time. Our simulations confirmed the success of the proposed approach, which results in relay motion control policies yielding significant performance improvement, when compared to agnostic, randomized relay motion.

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8 Appendices

8.1 Appendix A: Proofs / Section 3

8.1.1 Proof of Lemma 1

In the following, we will rely on an *incremental* construction of $\mathbf{\Sigma}$. Initially, consider the matrix

$$\widetilde{\mathbf{\Sigma}} \triangleq \begin{bmatrix} \widetilde{\mathbf{\Sigma}}(1,1) & \widetilde{\mathbf{\Sigma}}(1,2) & \dots & \widetilde{\mathbf{\Sigma}}(1,N_T) \\ \widetilde{\mathbf{\Sigma}}(2,1) & \widetilde{\mathbf{\Sigma}}(2,2) & \dots & \widetilde{\mathbf{\Sigma}}(2,N_T) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mathbf{\Sigma}}(N_T,1) & \widetilde{\mathbf{\Sigma}}(N_T,2) & \cdots & \widetilde{\mathbf{\Sigma}}(N_T,N_T) \end{bmatrix} \in \mathbb{S}^{RN_T}, \tag{8.1}$$

where, for each combination $(k,l) \in \mathbb{N}^+_{N_T} \times \mathbb{N}^+_{N_T}$, $\widetilde{\mathbf{\Sigma}}(k,l) \in \mathbb{S}^R$, with

$$\begin{aligned} \widetilde{\mathbf{\Sigma}}(k,l)(i,j) &\triangleq \widetilde{\mathbf{\Sigma}}\left(\mathbf{p}_i(k), \mathbf{p}_j(l)\right) \\ &\triangleq \eta^2 \exp\left(-\frac{\left\|\mathbf{p}_i(k) - \mathbf{p}_j(l)\right\|_2}{\beta}\right), \end{aligned} \tag{8.2}$$

for all $(i,j) \in \mathbb{N}^+_R \times \mathbb{N}^+_R$. By construction, $\widetilde{\mathbf{\Sigma}}$ is positive semidefinite, because the well known *exponential kernel* $\widetilde{\mathbf{\Sigma}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{++}$ defined above is positive (semi)definite.

Next, define the positive definite matrix

$$\mathbf{K} \triangleq \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}, \quad \text{with} \tag{8.3}$$

$$\kappa \triangleq \exp\left(-\frac{\|\mathbf{p}_S - \mathbf{p}_D\|_2}{\delta}\right) < 1 \tag{8.4}$$

and consider the *Tracy-Singh* type of product of $\mathbf{K}$ and $\widetilde{\mathbf{\Sigma}}$

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{K}} \triangleq \mathbf{K} \circ \widetilde{\mathbf{\Sigma}} \triangleq \begin{bmatrix} \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(1,1) & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(1,2) & \dots & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(1,N_T) \\ \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(2,1) & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(2,2) & \dots & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(2,N_T) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(N_T,1) & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(N_T,2) & \cdots & \mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(N_T,N_T) \end{bmatrix} \in \mathbb{S}^{2RN_T}, \tag{8.5}$$

where "$\otimes$" denotes the operator of the Kronecker product. Then, for each $(k,l) \in \mathbb{N}^+_{N_T} \times \mathbb{N}^+_{N_T}$, we have

$$\mathbf{K} \otimes \widetilde{\mathbf{\Sigma}}(k,l) \equiv \begin{bmatrix} \widetilde{\mathbf{\Sigma}}(k,l) & \kappa\widetilde{\mathbf{\Sigma}}(k,l) \\ \kappa\widetilde{\mathbf{\Sigma}}(k,l) & \widetilde{\mathbf{\Sigma}}(k,l) \end{bmatrix} \in \mathbb{S}^{2R}. \tag{8.6}$$

It is easy to show that $\widetilde{\mathbf{\Sigma}}_{\mathbf{K}}$ is positive semidefinite, that is, in $\mathbb{S}^{2RN_T}_+$. First, via a simple inductive argument, it can be shown that, for compatible matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$,

$$(\mathbf{AB}) \circ (\mathbf{CD}) \equiv (\mathbf{A} \circ \mathbf{C})(\mathbf{B} \circ \mathbf{D}). \tag{8.7}$$

Also, for compatible $\mathbf{A}, \mathbf{B}$, it is true that $(\mathbf{A} \circ \mathbf{B})^{\boldsymbol{T}} \equiv \mathbf{A}^{\boldsymbol{T}} \circ \mathbf{B}^{\boldsymbol{T}}$. Since $\mathbf{K}$ and $\widetilde{\mathbf{\Sigma}}$ are symmetric, consider their spectral decompositions $\mathbf{K} \equiv \mathbf{U}_{\mathbf{K}}\mathbf{\Lambda}_{\mathbf{K}}\mathbf{U}_{\mathbf{K}}^{\boldsymbol{T}}$ and $\widetilde{\mathbf{\Sigma}} \equiv \mathbf{U}_{\widetilde{\mathbf{\Sigma}}}\mathbf{\Lambda}_{\widetilde{\mathbf{\Sigma}}}\mathbf{U}_{\widetilde{\mathbf{\Sigma}}}^{\boldsymbol{T}}$. Given the identities stated above, we may write

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{K}} \equiv \mathbf{K} \circ \widetilde{\mathbf{\Sigma}} \equiv \left(\mathbf{U}_{\mathbf{K}}\mathbf{\Lambda}_{\mathbf{K}}\mathbf{U}_{\mathbf{K}}^{\boldsymbol{T}}\right) \circ \left(\mathbf{U}_{\widetilde{\mathbf{\Sigma}}}\mathbf{\Lambda}_{\widetilde{\mathbf{\Sigma}}}\mathbf{U}_{\widetilde{\mathbf{\Sigma}}}^{\boldsymbol{T}}\right)$$

$$
\begin{aligned}
&\equiv \left(\mathbf{U_K} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}\right)\left(\boldsymbol{\Lambda}_{\mathbf{K}} \circ \boldsymbol{\Lambda}_{\widetilde{\boldsymbol{\Sigma}}}\right)\left(\mathbf{U}_{\mathbf{K}}^{\boldsymbol{T}} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}^{\boldsymbol{T}}\right) \\
&\equiv \left(\mathbf{U_K} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}\right)\left(\boldsymbol{\Lambda}_{\mathbf{K}} \circ \boldsymbol{\Lambda}_{\widetilde{\boldsymbol{\Sigma}}}\right)\left(\mathbf{U_K} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}\right)^{\boldsymbol{T}},
\end{aligned} \tag{8.8}
$$

where $\left(\mathbf{U_K} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}\right)\left(\mathbf{U}_{\mathbf{K}}^{\boldsymbol{T}} \circ \mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}^{\boldsymbol{T}}\right) \equiv \left(\mathbf{U_K}\mathbf{U}_{\mathbf{K}}^{\boldsymbol{T}}\right) \circ \left(\mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}\mathbf{U}_{\widetilde{\boldsymbol{\Sigma}}}^{\boldsymbol{T}}\right) \equiv \mathbf{I}_2 \circ \mathbf{I}_{RN_T} \equiv \mathbf{I}_{2RN_T}$, and where the matrix $\boldsymbol{\Lambda}_{\mathbf{K}} \circ \boldsymbol{\Lambda}_{\widetilde{\boldsymbol{\Sigma}}}$ is easily shown to be diagonal and with nonnegative elements. Thus, since (8.8) constitutes a valid spectral decomposition for $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{K}}$, it follows that $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{K}} \in \mathbb{S}_{+}^{2RN_T}$.

As a last step, let $\mathbf{E} \in \mathbb{S}^{N_T}$, such that

$$
\mathbf{E}(k,l) \triangleq \exp\left(-\frac{|k-l|}{\gamma}\right), \tag{8.9}
$$

for all $(k,l) \in \mathbb{N}_{N_T}^{+} \times \mathbb{N}_{N_T}^{+}$. Again, $\mathbf{E}$ is positive semidefinite, because the well known *Laplacian kernel* is positive (semi)definite. Consider the matrix

$$
\widetilde{\boldsymbol{\Sigma}}_{\mathbf{E}} \triangleq \left(\mathbf{E} \otimes \mathbf{1}_{2R\times 2R}\right) \odot \widetilde{\boldsymbol{\Sigma}}_{\mathbf{K}} \in \mathbb{S}^{2RN_T}, \tag{8.10}
$$

where "$\odot$" denotes the operator of the Schur-Hadamard product. Of course, since the matrix $\mathbf{1}_{2R\times 2R}$ is rank-1 and positive semidefinite, $\mathbf{E} \otimes \mathbf{1}_{2R\times 2R}$ will be positive semidefinite as well. Consequently, by the Schur Product Theorem, $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{E}}$ will also be positive semidefinite. Finally, observe that

$$
\boldsymbol{\Sigma} \equiv \widetilde{\boldsymbol{\Sigma}}_{\mathbf{E}} + \sigma_{\xi}^2 \mathbf{I}_{2RN_T}, \tag{8.11}
$$

from where it follows that $\boldsymbol{\Sigma} \in \mathbb{S}_{++}^{2RN_T}$, whenever $\sigma_{\xi}^2 \neq 0$. Our claims follow. ■

8.1.2 Proof of Theorem 2

Obviously, the vector process $\boldsymbol{X}(t)$ is Gaussian with mean zero. This is straightforward to show. Therefore, what remains is, simply, to verify that the covariance structure of $\boldsymbol{X}(t)$ is the same as that of $\boldsymbol{C}(t)$, that is, we need to show that

$$
\mathbb{E}\left\{\boldsymbol{X}(s)\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} \equiv \mathbb{E}\left\{\boldsymbol{C}(s)\boldsymbol{C}^{\boldsymbol{T}}(t)\right\}, \tag{8.12}
$$

for all $(s,t) \in \mathbb{N}_{N_T}^{+} \times \mathbb{N}_{N_T}^{+}$.

First, consider the case where $s \equiv t$. Then, we have

$$
\begin{aligned}
\mathbb{E}\left\{\boldsymbol{X}(s)\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} &\equiv \mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} \\
&= \varphi^2 \mathbb{E}\left\{\boldsymbol{X}(t-1)\boldsymbol{X}^{\boldsymbol{T}}(t-1)\right\} + \left(1-\varphi^2\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}.
\end{aligned} \tag{8.13}
$$

Observe, though, that, similarly to the scalar order-1 autoregressive model, the quantity

$$
\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \equiv \mathbb{E}\left\{\boldsymbol{X}(0)\boldsymbol{X}^{\boldsymbol{T}}(0)\right\} \tag{8.14}
$$

is a fixed point of the previously stated recursion for $\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^{\boldsymbol{T}}(t)\right\}$. Therefore, it is true that

$$
\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} \equiv \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \equiv \boldsymbol{\Sigma}_{\boldsymbol{C}}(0) \equiv \mathbb{E}\left\{\boldsymbol{C}(t)\boldsymbol{C}^{\boldsymbol{T}}(t)\right\}, \tag{8.15}
$$

which the desired result.

Now, consider the case where $s < t$. Then, it may be easily shown that

$$\mathbb{E}\left\{\boldsymbol{X}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} \equiv \varphi^2\mathbb{E}\left\{\boldsymbol{X}(s-1)\,\boldsymbol{X}^{\boldsymbol{T}}(t-1)\right\} + \varphi\mathbb{E}\left\{\boldsymbol{W}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t-1)\right\}. \tag{8.16}$$

Let us consider the second term on the RHS of (8.16). Expanding the recursion, we may write

$$\begin{aligned}\varphi\mathbb{E}\left\{\boldsymbol{W}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t-1)\right\} &\equiv \varphi\mathbb{E}\left\{\boldsymbol{W}(s)\left(\varphi\boldsymbol{X}^{\boldsymbol{T}}(t-2) + \boldsymbol{W}^{\boldsymbol{T}}(t-1)\right)\right\} \\ &\equiv \varphi^2\mathbb{E}\left\{\boldsymbol{W}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t-2)\right\} \\ &\;\;\vdots \\ &\equiv \varphi^{t-s}\mathbb{E}\left\{\boldsymbol{W}(s)\,\boldsymbol{W}^{\boldsymbol{T}}(s)\right\} \\ &\equiv \varphi^{t-s}\left(1-\varphi^2\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}.\end{aligned} \tag{8.17}$$

We observe that this term depends only on the lag $t-s$. Thus, it is true that

$$\begin{aligned}\mathbb{E}\left\{\boldsymbol{X}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} &\equiv \varphi^2\mathbb{E}\left\{\boldsymbol{X}(s-1)\,\boldsymbol{X}^{\boldsymbol{T}}(t-1)\right\} + \varphi^{t-s}\left(1-\varphi^2\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \\ &= \varphi^{2\cdot 2}\mathbb{E}\left\{\boldsymbol{X}(s-2)\,\boldsymbol{X}^{\boldsymbol{T}}(t-2)\right\} + \varphi^{t-s}\left(1-\varphi^2\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}\left(1+\varphi^2\right) \\ &\;\;\vdots \\ &\equiv \varphi^{2s}\mathbb{E}\left\{\boldsymbol{X}(0)\,\boldsymbol{X}^{\boldsymbol{T}}(t-s)\right\} + \varphi^{t-s}\left(1-\varphi^2\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}\sum_{i\in\mathbb{N}_{s-1}}\left(\varphi^2\right)^{s-1} \\ &= \varphi^{2s}\mathbb{E}\left\{\boldsymbol{X}(0)\,\boldsymbol{X}^{\boldsymbol{T}}(t-s)\right\} + \varphi^{t-s}\left(1-\varphi^{2s}\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}.\end{aligned} \tag{8.18}$$

Further, we may expand $\mathbb{E}\left\{\boldsymbol{X}(0)\,\boldsymbol{X}^{\boldsymbol{T}}(t-s)\right\}$ in similar fashion as above, to get that

$$\mathbb{E}\left\{\boldsymbol{X}(0)\,\boldsymbol{X}^{\boldsymbol{T}}(t-s)\right\} \equiv \varphi^{t-s}\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}}. \tag{8.19}$$

Exactly the same arguments may be made for the symmetric case where $t < s$. Therefore, it follows that

$$\begin{aligned}\mathbb{E}\left\{\boldsymbol{X}(s)\,\boldsymbol{X}^{\boldsymbol{T}}(t)\right\} &\equiv \varphi^{|t-s|}\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \\ &\equiv \exp\left(-\frac{|t-s|}{\gamma}\right)\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{C}} \\ &\equiv \boldsymbol{\Sigma}_{\boldsymbol{C}}(t-s)\end{aligned} \tag{8.20}$$

for all $(s,t) \in \mathbb{N}^+_{N_T} \times \mathbb{N}^+_{N_T}$, and we are done. ■

8.2 Appendix B: Measurability & The Fundamental Lemma of Stochastic Control

In the following, aligned with the purposes of this paper, a detailed discussion is presented, which is related to important technical issues, arising towards the analysis and simplification of variational problems of the form of (4.5).

At this point, it would be necessary to introduce some important concepts. Let us first introduce the useful class of *Carathéodory functions* [26,44][2].

Definition 3. (Carathéodory Function [26,44]) On $(\Omega, \mathscr{F})$, the mapping $H : \Omega \times \mathbb{R}^N \rightarrow \overline{\mathbb{R}}$ is called Carathéodory, if and only if $H(\cdot, \boldsymbol{x})$ is $\mathscr{F}$-measurable for all $\boldsymbol{x} \in \mathbb{R}^N$ and $H(\omega, \cdot)$ is continuous for all $\omega \in \Omega$.

Remark 8. As the reader might have already observed, Carathéodory functions and random fields with (everywhere) continuous sample paths are essentially the same thing. Nevertheless, the term "Carathéodory function" is extensively used in our references [25,26,44]. This is the main reason why we still define and use the term. ■

In the analysis that follows, we will exploit the notion of *measurability* for closed-valued multifunctions.

Definition 4. (Measurable Multifunctions [25,26]) On the measurable space $(\Omega, \mathscr{F})$, a closed-valued multifunction $\mathcal{X} : \Omega \rightrightarrows \mathbb{R}^N$ is $\mathscr{F}$-measurable if and only if, for all closed $\mathcal{A} \subseteq \mathbb{R}^N$, the preimage

$$\mathcal{X}^{-1}(\mathcal{A}) \triangleq \left\{ \omega \in \Omega \middle| \mathcal{X}(\omega) \bigcap \mathcal{A} \neq \varnothing \right\} \tag{8.21}$$

is in $\mathscr{F}$. If $\mathscr{F}$ constitutes a Borel σ-algebra, generated by a topology on Ω, then an $\mathscr{F}$-measurable $\mathcal{X}$ will be equivalently called *Borel measurable*.

We will also make use of the concept of a *closed multifunction* (Remark 28 in [26], p. 365), whose definition is also presented below, restricted to the case of Euclidean spaces, of interest in this work.

Definition 5. (Closed Multifunction [26]) A closed-valued multifunction $\mathcal{X} : \mathbb{R}^M \rightrightarrows \mathbb{R}^N$ (a function from $\mathbb{R}^M$ to closed sets in $\mathbb{R}^N$) is closed if and only if, for all sequences $\{\boldsymbol{x}_k\}_{k \in \mathbb{N}}$ and $\{\boldsymbol{y}_k\}_{k \in \mathbb{N}}$, such that $\boldsymbol{x}_k \underset{k \rightarrow \infty}{\longrightarrow} \boldsymbol{x}$, $\boldsymbol{y}_k \underset{k \rightarrow \infty}{\longrightarrow} \boldsymbol{y}$ and $\boldsymbol{x}_k \in \mathcal{X}(\boldsymbol{y}_k)$, for all $k \in \mathbb{N}$, it is true that $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{y})$.

8.2.1 Random Functions & The Substitution Rule for Conditional Expectations

Given a random function $g(\omega, \boldsymbol{x})$, a sub σ-algebra $\mathscr{Y}$, another $\mathscr{Y}$-measurable random element X, and as long as $\mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\}$ exists for all $\boldsymbol{x}$ in the range of X, we would also need to make extensive use of the *substitution rule*

$$\begin{aligned} \mathbb{E}\{g(\cdot, X) | \mathscr{Y}\}(\omega) &\equiv \mathbb{E}\{g(\cdot, X(\omega)) | \mathscr{Y}\}(\omega) \\ &\equiv \mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\}(\omega)|_{\boldsymbol{x} \equiv X(\omega)}, \quad \mathcal{P} - a.e., \end{aligned} \tag{8.22}$$

which would allow us to evaluate conditional expectations, by essentially fixing the quantities that are constant relative to the information we are conditioning on, carry out the evaluation, and then let those quantities vary in ω again. Although the substitution rule is a concept readily taken for granted when conditional expectations of Borel measurable functions of random elements (say, from products of Euclidean spaces to $\mathbb{R}$) are considered, it does not hold, in general, for arbitrary random functions. As far as our general formulation is concerned, it is necessary to consider random

[2] Instead of working with the class of Carathéodory functions, we could also consider the more general class of *random lower semicontinuous functions* [26], which includes the former. However, this might lead to overgeneralization and, thus, we prefer not to do so; the class of Carathéodory functions will be perfectly sufficient for our purposes.

functions, whose domain is a product of a well behaved space (such as $\mathbb{R}^N$) and the sample space, Ω, whose structure is assumed to be and should be arbitrary, at least in regard to the applications of interest in this work.

One common way to ascertain the validity of the substitution rule is by exploitting the representation of conditional expectations via integrals with respect to the relevant regular conditional distributions, whenever the latter exist. But because of the arbitrary structure of the base space $(\Omega, \mathscr{F}, \mathcal{P})$, regular conditional distributions defined on points in the sample space Ω cannot be guaranteed to exist and, therefore, the substitution rule may fail to hold. However, as we will see, the substitution rule will be very important for establishing the Fundamental Lemma. Therefore, we may choose to impose it as a property on the structures of g and/or X instead, as well as establish sufficient conditions for this property to hold. The relevant definition follows.

Definition 6. (Substitution Property (*SP*)) On $(\Omega, \mathscr{F}, \mathcal{P})$, consider a random element $Y : \Omega \to \mathbb{R}^M$, the associated sub σ-algebra $\mathscr{Y} \triangleq \sigma\{Y\} \subseteq \mathscr{F}$, and a random function $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, such that $\mathbb{E}\{g(\cdot, \boldsymbol{x})\}$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$. Let $\mathfrak{C}_{\mathscr{Y}}$ be any functional class, such that[3]

$$\mathfrak{C}_{\mathscr{Y}} \subseteq \mathfrak{I}_{\mathscr{Y}} \triangleq \left\{ X : \Omega \to \mathbb{R}^N \middle| \begin{array}{c} X^{-1}(\mathcal{A}) \in \mathscr{Y}, \text{ for all } \mathcal{A} \in \mathscr{B}\left(\mathbb{R}^N\right) \\ \mathbb{E}\{g(\cdot, X)\} \text{ exists} \end{array} \right\}. \tag{8.23}$$

We say that g *possesses the Substitution Property within* $\mathfrak{C}_{\mathscr{Y}}$, or, equivalently, that g *is* $\boldsymbol{SP}\Diamond\mathfrak{C}_{\mathscr{Y}}$, if and only if there exists a jointly Borel measurable function $h : \mathbb{R}^M \times \mathbb{R}^N \to \overline{\mathbb{R}}$, with $h(Y(\omega), \boldsymbol{x}) \equiv \mathbb{E}\{ g(\cdot, \boldsymbol{x})| \mathscr{Y}\}(\omega)$, everywhere in $(\omega, \boldsymbol{x}) \in \Omega \times \mathbb{R}^N$, such that, for any $X \in \mathfrak{C}_{\mathscr{Y}}$, it is true that

$$\mathbb{E}\{ g(\cdot, X)| \mathscr{Y}\}(\omega) \equiv h(Y(\omega), X(\omega)), \tag{8.24}$$

almost everywhere in $\omega \in \Omega$ with respect to $\mathcal{P}$.

Remark 9. Observe that, in Definition 6, h is required to be the same for all $X \in \mathfrak{C}_{\mathscr{Y}}$. That is, h should be determined only by the structure of g, relative to $\mathscr{Y}$, regardless of the specific X within $\mathfrak{C}_{\mathscr{Y}}$, considered each time. On the other hand, it is also important to note that the set of unity measure, where (8.24) is valid, *might indeed be dependent on the particular* X. ■

Remark 10. Another detail of Definition 6 is that, because $\mathbb{E}\{g(\cdot, \boldsymbol{x})\}$ is assumed to exist for all $\boldsymbol{x} \in \mathbb{R}^N$, $\mathbb{E}\{ g(\cdot, \boldsymbol{x})| \mathscr{Y}\}$ also exists and, as an extended $\mathscr{Y}$-measurable random variable, for every $\boldsymbol{x} \in \mathbb{R}^N$, there exists a Borel measurable function $h_{\boldsymbol{x}} : \mathbb{R}^M \to \overline{\mathbb{R}}$, such that

$$h_{\boldsymbol{x}}(Y(\omega)) \equiv \mathbb{E}\{ g(\cdot, \boldsymbol{x})| \mathscr{Y}\}(\omega), \quad \forall \omega \in \Omega. \tag{8.25}$$

One may then readily define a function $h : \mathbb{R}^M \times \mathbb{R}^N \to \overline{\mathbb{R}}$, such that $h(Y(\omega), \boldsymbol{x}) \equiv \mathbb{E}\{ g(\cdot, \boldsymbol{x})| \mathscr{Y}\}(\omega)$, uniformly ***for all points***, ω, ***of the sample space***, Ω. This is an extremely important fact, in regard to the analysis that follows. Observe, however, that, in general, h will be Borel measurable *only in its first argument*; h is not guaranteed to be measurable in $\boldsymbol{x} \in \mathbb{R}^N$, for each $Y \in \mathbb{R}^M$, let alone jointly measurable in both its arguments. ■

Remark 11. **(Generalized *SP*)** Definition 6 may be reformulated in a more general setting. In particular, $\mathscr{Y}$ may be assumed to be any arbitrary sub σ-algebra of $\mathscr{F}$, but with the subtle difference that, in such case, one would instead directly demand that the random function $h : \Omega \times \mathbb{R}^N \to \overline{\mathbb{R}}$, with

[3]Hereafter, statements of type "$\mathbb{E}\{g(\cdot, X)\}$ exists" will *implicitly* imply that $g(\cdot, X)$ is an $\mathscr{F}$-measurable function.

$h(\omega,\boldsymbol{x}) \equiv \mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}(\omega)$, everywhere in $(\omega,\boldsymbol{x}) \in \Omega \times \mathbb{R}^N$, is jointly $\mathscr{Y} \otimes \mathscr{B}\left(\mathbb{R}^N\right)$-measurable and such that, for any $X \in \mathfrak{C}_{\mathscr{Y}}$ (with $\mathfrak{C}_{\mathscr{Y}}$ defined accordingly), it is true that

$$\mathbb{E}\{g(\cdot,X)|\mathscr{Y}\}(\omega) \equiv h(\omega,X(\omega)), \quad \mathcal{P}-a.e.. \tag{8.26}$$

Although such a generalized definition of the substitution property is certainly less enlightening, it is still useful. Specifically, this version of $\boldsymbol{SP}$ is explicitly used in the statement and proof of Theorem 6, presented in Section 4.4. ■

Keeping $(\Omega,\mathscr{F},\mathcal{P})$ of arbitrary structure, we will be interested in the set of g's which are $\boldsymbol{SP}\diamondsuit\mathfrak{I}_{\mathscr{Y}}$. The next result provides a large class of such random functions, which is sufficient for our purposes.

Theorem 8. (Sufficient Conditions for the $\boldsymbol{SP}\diamondsuit\mathfrak{I}_{\mathscr{Y}}$) *On $(\Omega,\mathscr{F},\mathcal{P})$, consider a random element $Y:\Omega\to\mathbb{R}^M$, the associated sub σ-algebra $\mathscr{Y} \triangleq \sigma\{Y\} \subseteq \mathscr{F}$, and a random function $g:\Omega\times\mathbb{R}^N\to\mathbb{R}$. Suppose that:*

- *g is dominated by a $\mathcal{P}$-integrable function; that is,*

$$\exists \psi \in \mathcal{L}_1(\Omega,\mathscr{F},\mathcal{P};\mathbb{R}), \text{ such that } \sup_{\boldsymbol{x}\in\mathbb{R}^N} |g(\omega,\boldsymbol{x})| \le \psi(\omega), \quad \forall \omega \in \Omega, \tag{8.27}$$

- *g is Carathéodory on $\Omega\times\mathbb{R}^N$, and that*
- *the extended real valued function $\mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}$ is Carathéodory on $\Omega\times\mathbb{R}^N$.*

Then, g is $\boldsymbol{SP}\diamondsuit\mathfrak{I}_{\mathscr{Y}}$.

Proof of Theorem 8. Under the setting of the theorem, consider any $\mathscr{Y}$-measurable random element $X:\Omega\to\mathbb{R}^N$, for which $\mathbb{E}\{g(\cdot,X)\}$ exists. Then, $\mathbb{E}\{g(\cdot,X)|\mathscr{Y}\}$ exists. Also, by domination of g by ψ, for all $\boldsymbol{x}\in\mathbb{R}^N$, $\mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}$ exists and constitutes a $\mathcal{P}$-integrable, $\mathscr{Y}$-measurable random variable. By Remark 10, we know that

$$\mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}(\omega) \equiv h(Y(\omega),\boldsymbol{x}), \quad \forall(\omega,\boldsymbol{x}) \in \Omega\times\mathbb{R}^N, \tag{8.28}$$

where $h:\mathbb{R}^M\times\mathbb{R}^N\to\overline{\mathbb{R}}$ is Borel measurable *in its first argument*. However, since $\mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}(\omega) \equiv h(Y(\omega),\boldsymbol{x})$ is Carathéodory on $\Omega\times\mathbb{R}^N$, h is Carathéodory on $\mathbb{R}^M\times\mathbb{R}^N$, as well. Thus, h will be jointly $\mathscr{B}\left(\mathbb{R}^M\right)\otimes\mathscr{B}\left(\mathbb{R}^N\right)$-measurable (Lemma 4.51 in [44], along with the fact that $\overline{\mathbb{R}}$ is metrizable).

We claim that, actually, h is such that

$$\mathbb{E}\{g(\cdot,X)|\mathscr{Y}\} \equiv h(Y,X), \quad \mathcal{P}-a.e.. \tag{8.29}$$

Employing a common technique, the result will be proven in steps, starting from indicators and building up to arbitrary measurable functions, as far as X is concerned. Before embarking with the core of the proof, note that, for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $\mathbb{R}^N$ and any $\mathcal{A}\in\mathscr{F}$, the sum $g(\cdot,\boldsymbol{x}_1)\mathbb{1}_{\mathcal{A}} + g(\cdot,\boldsymbol{x}_2)\mathbb{1}_{\mathcal{A}^c}$ is always well defined, and $\mathbb{E}\{g(\cdot,\boldsymbol{x}_1)\mathbb{1}_{\mathcal{A}}\}$ and $\mathbb{E}\{g(\cdot,\boldsymbol{x}_2)\mathbb{1}_{\mathcal{A}^c}\}$ both exist and are finite by domination. This implies that $\mathbb{E}\{g(\cdot,\boldsymbol{x}_1)\mathbb{1}_{\mathcal{A}}\}+\mathbb{E}\{g(\cdot,\boldsymbol{x}_2)\mathbb{1}_{\mathcal{A}^c}\}$ is always well-defined, which in turn implies the validity of the additivity properties (Theorem 1.6.3 and Theorem 5.5.2 in [45])

$$\mathbb{E}\{g(\cdot,\boldsymbol{x}_1)\mathbb{1}_{\mathcal{A}} + g(\cdot,\boldsymbol{x}_2)\mathbb{1}_{\mathcal{A}^c}\} \equiv \mathbb{E}\{g(\cdot,\boldsymbol{x}_1)\mathbb{1}_{\mathcal{A}}\} + \mathbb{E}\{g(\cdot,\boldsymbol{x}_2)\mathbb{1}_{\mathcal{A}^c}\} \in \mathbb{R}, \quad \text{and} \tag{8.30}$$

$$\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}_1\right)\mathbb{1}_{\mathcal{A}}+g\left(\cdot,\boldsymbol{x}_2\right)\mathbb{1}_{\mathcal{A}^c}\middle|\mathscr{Y}\right\}\equiv\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}_1\right)\mathbb{1}_{\mathcal{A}}\middle|\mathscr{Y}\right\}+\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}_2\right)\mathbb{1}_{\mathcal{A}^c}\middle|\mathscr{Y}\right\},\mathcal{P}-a.e.. \quad (8.31)$$

Hence, under our setting, any such manipulation is technically justified.

Suppose first that $X(\omega)\equiv\widetilde{\boldsymbol{x}}\mathbb{1}_{\mathcal{A}}(\omega)$, for some $\widetilde{\boldsymbol{x}}\in\mathbb{R}^N$ and some $\mathcal{A}\in\mathscr{Y}$. Then, by ([45], Theorem 5.5.11 & Comment 5.5.12), it is true that

$$\begin{aligned}\mathbb{E}\left\{ g\left(\cdot,X\right)\middle|\mathscr{Y}\right\} &\equiv\mathbb{E}\left\{ g\left(\cdot,\widetilde{\boldsymbol{x}}\right)\mathbb{1}_{\mathcal{A}}+g\left(\cdot,\boldsymbol{0}\right)\mathbb{1}_{\mathcal{A}^c}\middle|\mathscr{Y}\right\}\\ &\equiv\mathbb{E}\left\{ g\left(\cdot,\widetilde{\boldsymbol{x}}\right)\mathbb{1}_{\mathcal{A}}\middle|\mathscr{Y}\right\}+\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{0}\right)\mathbb{1}_{\mathcal{A}^c}\middle|\mathscr{Y}\right\}\\ &\equiv\mathbb{1}_{\mathcal{A}}\mathbb{E}\left\{ g\left(\cdot,\widetilde{\boldsymbol{x}}\right)\middle|\mathscr{Y}\right\}+\mathbb{1}_{\mathcal{A}^c}\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{0}\right)\middle|\mathscr{Y}\right\}\\ &\equiv\mathbb{1}_{\mathcal{A}}h\left(Y,\widetilde{\boldsymbol{x}}\right)+\mathbb{1}_{\mathcal{A}^c}h\left(Y,\boldsymbol{0}\right)\\ &\equiv h\left(Y,\widetilde{\boldsymbol{x}}\mathbb{1}_{\mathcal{A}}\right)\\ &\equiv h\left(Y,X\right),\quad\mathcal{P}-a.e.,\end{aligned}\qquad(8.32)$$

proving the claim for indicators.

Consider now simple functions of the form

$$X(\omega)\equiv\sum_{i\in\mathbb{N}_I^+}\widetilde{\boldsymbol{x}}_i\mathbb{1}_{\mathcal{A}_i}(\omega),\qquad(8.33)$$

where $\widetilde{\boldsymbol{x}}_i\in\mathbb{R}^N$, $\mathcal{A}_i\in\mathscr{Y}$, for all $i\in\mathbb{N}_I^+$, with $\mathcal{A}_i\bigcap\mathcal{A}_j\equiv\varnothing$, for $i\neq j$ and $\bigcup_{i\in\mathbb{N}_I^+}\mathcal{A}_i\equiv\Omega$. Then, we again have

$$\begin{aligned}\mathbb{E}\left\{ g\left(\cdot,X\right)\middle|\mathscr{Y}\right\} &\equiv\mathbb{E}\left\{ \sum_{i\in\mathbb{N}_I^+}g\left(\cdot,\widetilde{\boldsymbol{x}}_i\right)\mathbb{1}_{\mathcal{A}_i}\middle|\mathscr{Y}\right\}\\ &\equiv\sum_{i\in\mathbb{N}_I^+}\mathbb{E}\left\{ g\left(\cdot,\widetilde{\boldsymbol{x}}_i\right)\mathbb{1}_{\mathcal{A}_i}\middle|\mathscr{Y}\right\}\\ &\equiv\sum_{i\in\mathbb{N}_I^+}\mathbb{1}_{\mathcal{A}_i}\mathbb{E}\left\{ g\left(\cdot,\widetilde{\boldsymbol{x}}_i\right)\middle|\mathscr{Y}\right\}\\ &\equiv\sum_{i\in\mathbb{N}_I^+}\mathbb{1}_{\mathcal{A}_i}h\left(Y,\widetilde{\boldsymbol{x}}_i\right)\\ &\equiv h\left(Y,\sum_{i\in\mathbb{N}_I^+}\widetilde{\boldsymbol{x}}_i\mathbb{1}_{\mathcal{A}_i}\right)\\ &\equiv h\left(Y,X\right),\quad\mathcal{P}-a.e.,\end{aligned}\qquad(8.34)$$

and the proved is claimed for simple functions.

To show that our claims are true for any arbitrary random function g, we take advantage of the continuity of both h and g in $\boldsymbol{x}$. First, we know that h is Carathéodory, which means that, for every $\omega\in\Omega$, if any sequence $\left\{\boldsymbol{x}_n\in\mathbb{R}^N\right\}_{n\in\mathbb{N}}$ is such that $\boldsymbol{x}_n\underset{n\to\infty}{\longrightarrow}\boldsymbol{x}$ (for arbitrary $\boldsymbol{x}\in\mathbb{R}^N$), it is true that

$$h\left(Y(\omega),\boldsymbol{x}_n\right)\equiv\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}_n\right)\middle|\mathscr{Y}\right\}(\omega)\underset{n\to\infty}{\longrightarrow}\mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}\right)\middle|\mathscr{Y}\right\}(\omega)\equiv h\left(Y(\omega),\boldsymbol{x}\right).\qquad(8.35)$$

Second, we know that g is Carathéodory as well, also implying that, for every $\omega \in \Omega$, if any sequence $\left\{\boldsymbol{x}_n \in \mathbb{R}^N\right\}_{n\in\mathbb{N}}$ is such that $\boldsymbol{x}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{x}$, it is true that

$$g\left(\omega, \boldsymbol{x}_n\right) \underset{n\to\infty}{\longrightarrow} g\left(\omega, \boldsymbol{x}\right). \tag{8.36}$$

Next, let $\left\{X_n : \Omega \to \mathbb{R}^N\right\}_{n\in\mathbb{N}}$ be a sequence of simple Borel functions, such that, for all $\omega \in \Omega$,

$$X_n\left(\omega\right) \underset{n\to\infty}{\longrightarrow} X\left(\omega\right). \tag{8.37}$$

Note that such a sequence always exists (see Theorem 1.5.5 (b) in [45]). Consequently, for each $\omega \in \Omega$, we may write (note that g is $\mathscr{F} \otimes \mathscr{B}\left(\mathbb{R}^N\right)$-measurable; see ([44], Lemma 4.51))

$$g\left(\omega, X_n\left(\omega\right)\right) \underset{n\to\infty}{\longrightarrow} g\left(\omega, X\left(\omega\right)\right), \tag{8.38}$$

that is, the sequence $\left\{g\left(\cdot, X_n\right)\right\}_{n\in\mathbb{N}}$ converges to $g\left(\cdot, X\right)$, everywhere in Ω.

Now, let us try to apply the Dominated Convergence Theorem for conditional expectations (Theorem 5.5.5 in [45]) to the aforementioned sequence of functions. Of course, we have to show that all members of the sequence $\left\{g\left(\cdot, X_n\right)\right\}_{n\in\mathbb{N}}$ are dominated by another integrable function, uniformly in $n \in \mathbb{N}$. By assumption, there exists an integrable function $\psi : \Omega \to \mathbb{R}$, such that

$$\left|g\left(\omega, \boldsymbol{x}\right)\right| \le \psi\left(\omega\right), \quad \forall \left(\omega, \boldsymbol{x}\right) \in \Omega \times \mathbb{R}^N. \tag{8.39}$$

In particular, it must also be true that

$$\left|g\left(\omega, X_n\left(\omega\right)\right)\right| \le \psi\left(\omega\right), \quad \forall \left(\omega, n\right) \in \Omega \times \mathbb{N}, \tag{8.40}$$

verifying the domination requirement. Thus, Dominated Convergence implies the existence of an event $\Omega_{\Pi_1} \subseteq \Omega$, with $\mathcal{P}\left(\Omega_{\Pi_1}\right) \equiv 1$, such that, for all $\omega \in \Omega_{\Pi_1}$,

$$\mathbb{E}\left\{\left. g\left(\cdot, X_n\right)\right| \mathscr{Y}\right\}\left(\omega\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left\{\left. g\left(\cdot, X\right)\right| \mathscr{Y}\right\}\left(\omega\right). \tag{8.41}$$

Also, for every $\omega \in \Omega \bigcap \Omega_{\Pi_1} \equiv \Omega_{\Pi_1}$, (8.35) yields

$$h\left(Y\left(\omega\right), X_n\left(\omega\right)\right) \underset{n\to\infty}{\longrightarrow} h\left(Y\left(\omega\right), X\left(\omega\right)\right). \tag{8.42}$$

However, by what we have shown above, because the sequence $\left\{X_n\right\}_{n\in\mathbb{N}}$ consists of simple functions, then, for every $n \in \mathbb{N}$, there exists $\Omega_{\Pi^n} \subseteq \Omega$, with $\mathcal{P}\left(\Omega_{\Pi^n}\right) \equiv 1$, such that, for all $\omega \in \Omega_{\Pi^n}$,

$$\mathbb{E}\left\{\left. g\left(\cdot, X_n\right)\right| \mathscr{Y}\right\}\left(\omega\right) \equiv h\left(Y\left(\omega\right), X_n\left(\omega\right)\right). \tag{8.43}$$

Since $\mathbb{N}$ is countable, there exists a "global" event $\Omega_{\Pi_2} \subseteq \Omega$, with $\mathcal{P}\left(\Omega_{\Pi_2}\right) \equiv 1$, such that, for all $\omega \in \Omega_{\Pi_2}$,

$$\mathbb{E}\left\{\left. g\left(\cdot, X_n\right)\right| \mathscr{Y}\right\}\left(\omega\right) \equiv h\left(Y\left(\omega\right), X_n\left(\omega\right)\right), \quad \forall n \in \mathbb{N}. \tag{8.44}$$

Now define the event $\Omega_{\Pi_3} \triangleq \Omega_{\Pi_1} \bigcap \Omega_{\Pi_2}$. Of course, $\mathcal{P}\left(\Omega_{\Pi_3}\right) \equiv 1$. Then, for every $\omega \in \Omega_{\Pi_3}$, (8.41), (8.42) and (8.44) all hold simultaneously. Therefore, for every $\omega \in \Omega_{\Pi_3}$, it is true that (say)

$$h\left(Y\left(\omega\right), X_n\left(\omega\right)\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left\{\left. g\left(\cdot, X\right)\right| \mathscr{Y}\right\}\left(\omega\right) \quad \text{and} \tag{8.45}$$

$$h\left(Y\left(\omega\right),X_n\left(\omega\right)\right)\underset{n\to\infty}{\longrightarrow}h\left(Y\left(\omega\right),X\left(\omega\right)\right), \tag{8.46}$$

which immediately yields

$$\mathbb{E}\left\{\left.g\left(\cdot,X\right)\right|\mathscr{Y}\right\}\left(\omega\right)\equiv h\left(Y\left(\omega\right),X\left(\omega\right)\right),\quad \mathcal{P}-a.e., \tag{8.47}$$

showing that g is $\boldsymbol{SP}\Diamond\mathfrak{I}_{\mathscr{Y}}$. ■

Remark 12. We would like to note that the assumptions of Theorem 8 can be significantly weakened, guaranteeing the validity of the substitution rule for vastly discontinuous random functions, including, for instance, cases with random discontinuities, or random jumps. This extended analysis, though, is out of the scope of the paper and will be presented elsewhere. ■

8.2.2 A Base Form of the Lemma

We will first state a base, very versatile version of the Fundamental Lemma, treating a general class of problems, which includes the particular stochastic problem of interest, (4.5), as a subcase.

Lemma 3. (Fundamental Lemma / Base Version) *On $(\Omega,\mathscr{F},\mathcal{P})$, consider a random element $Y:\Omega\to\mathbb{R}^M$, the sub σ-algebra $\mathscr{Y}\triangleq\sigma\{Y\}\subseteq\mathscr{F}$, a random function $g:\Omega\times\mathbb{R}^N\to\mathbb{R}$, such that $\mathbb{E}\{g(\cdot,\boldsymbol{x})\}$ exists for all $\boldsymbol{x}\in\mathbb{R}^N$, a Borel measurable closed-valued multifunction $\mathcal{X}:\mathbb{R}^N\rightrightarrows\mathbb{R}^N$, with $\operatorname{dom}(\mathcal{X})\equiv\mathbb{R}^N$, as well as another $\mathscr{Y}$-measurable random element $Z_Y:\Omega\to\mathbb{R}^N$, with $Z_Y(\omega)\equiv\mathcal{Z}(Y(\omega))$, for all $\omega\in\Omega$, for some Borel $\mathcal{Z}:\mathbb{R}^M\to\mathbb{R}^N$. Consider also the decision set*

$$\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}\triangleq\left\{X:\Omega\to\mathbb{R}^N\left|\begin{array}{l}X\left(\omega\right)\in\mathcal{X}\left(Z_Y\left(\omega\right)\right),\ a.e.\ in\ \omega\in\Omega\\ X^{-1}\left(\mathcal{A}\right)\in\mathscr{Y},\ for\ all\ \mathcal{A}\in\mathscr{B}\left(\mathbb{R}^N\right)\end{array}\right.\right\}, \tag{8.48}$$

containing all $\mathscr{Y}$-measurable selections of $\mathcal{X}(Z_Y)$. Then, $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$ is nonempty. Suppose that:

- *$\mathbb{E}\{g(\cdot,X)\}$ exists for all $X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, with $\inf_{X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}}\mathbb{E}\{g(\cdot,X)\}<+\infty$, and that*
- *g is $\boldsymbol{SP}\Diamond\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$.*

Then, if $\overline{\mathscr{Y}}$ denotes the completion of $\mathscr{Y}$ relative to the restriction $\mathcal{P}|_{\mathscr{Y}}$, then the optimal value function $\inf_{\boldsymbol{x}\in\mathcal{X}(Z_Y)}\mathbb{E}\{g(\cdot,\boldsymbol{x})|\mathscr{Y}\}\triangleq\vartheta$ is $\overline{\mathscr{Y}}$-measurable and it is true that

$$\inf_{X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}}\mathbb{E}\left\{g\left(\cdot,X\right)\right\}\equiv\mathbb{E}\left\{\inf_{\boldsymbol{x}\in\mathcal{X}(Z_Y)}\mathbb{E}\left\{\left.g\left(\cdot,\boldsymbol{x}\right)\right|\mathscr{Y}\right\}\right\}\equiv\mathbb{E}\left\{\vartheta\right\}. \tag{8.49}$$

In other words, variational minimization over $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$ is exchangeable by pointwise (over constants) minimization over the random multifunction $\mathcal{X}(Z_Y)$, relative to $\mathscr{Y}$.

Remark 13. Note that, in the statement of Lemma 3, assuming that the infimum of $\mathbb{E}\{g(\cdot,X)\}$ over $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$ is less than $+\infty$ is *equivalent* to assuming the existence of an X in $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, such that $\mathbb{E}\{g(\cdot,X)\}$ is less than $+\infty$. ■

Before embarking with the proof of Lemma 3, it would be necessary to state an old, fundamental selection theorem, due to Mackey [46].

Theorem 9. (Borel Measurable Selections [46]**)** *Let $(\mathcal{S}_1, \mathscr{B}(\mathcal{S}_1))$ and $(\mathcal{S}_2, \mathscr{B}(\mathcal{S}_2))$ be Borel spaces and let $(\mathcal{S}_2, \mathscr{B}(\mathcal{S}_2))$ be standard. Let $\mu : \mathscr{B}(\mathcal{S}_1) \to [0, \infty]$ be a standard measure on $(\mathcal{S}_1, \mathscr{B}(\mathcal{S}_1))$. Suppose that $\mathcal{A} \in \mathscr{B}(\mathcal{S}_1) \otimes \mathscr{B}(\mathcal{S}_2)$, such that, for each $y \in \mathcal{S}_1$, there exists $x_y \in \mathcal{S}_2$, so that $(y, x_y) \in \mathcal{A}$. Then, there exists a Borel subset $\mathcal{O} \in \mathscr{B}(\mathcal{S}_1)$ with $\mu(\mathcal{O}) \equiv 0$, as well as a Borel measurable function $\phi : \mathcal{S}_1 \to \mathcal{S}_2$, such that $(y, \phi(y)) \in \mathcal{A}$, for all $y \in \mathcal{S}_1 \setminus \mathcal{O}$.*

Remark 14. Theorem 9 refers to the concepts of a *Borel space,* a *standard Borel space* and a *standard measure*. These are employed as structural assumptions, in order for the conclusions of the theorem to hold true. In this paper, except for the base probability space $(\Omega, \mathscr{F}, \mathcal{P})$, whose structure may be arbitrary, all other spaces and measures considered will satisfy those assumptions by default. We thus choose not to present the respective definitions; instead, the interested reader is referred to the original article, [46]. ■

We are now ready to prove Lemma 3, as follows.

Proof of Lemma 3. As usual with such results, the proof will rely on showing a double sided inequality [23–25, 27, 28, 47]. There is one major difficulty, though, in the optimization setting considered, because all infima may be potentially unattainable, within the respective decision sets. However, it is immediately evident that, because g is assumed to be $\boldsymbol{SP}\Diamond\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, and via a simple application of the tower property, it will suffice to show that

$$\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y, X)\} \equiv \mathbb{E}\left\{\inf_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} h(Y, \boldsymbol{x})\right\}. \tag{8.50}$$

This is because it is true that, for any $\mathscr{Y}$-measurable selection of $\mathcal{X}(Z_Y)$, say $X : \Omega \to \mathbb{R}^N$, for which $\mathbb{E}\{g(\cdot, X)\}$ exists,

$$\mathbb{E}\{g(\cdot, X)|\, \mathscr{Y}\}(\omega) \equiv h(Y(\omega), \boldsymbol{x})|_{\boldsymbol{x}=X(\omega)}, \quad \forall \omega \in \Omega_{\Pi_X}, \tag{8.51}$$

where the event $\Omega_{\Pi_X} \in \mathscr{F}$ is such that $\mathcal{P}\left(\Omega_{\Pi_X}\right) \equiv 1$ and h is jointly Borel, satisfying

$$\mathbb{E}\{g(\cdot, \boldsymbol{x})|\, \mathscr{Y}\} \equiv h(Y(\omega), \boldsymbol{x}), \tag{8.52}$$

everywhere in $(\omega, \boldsymbol{x}) \in \Omega \times \mathbb{R}^N$.

For the sake of clarity in the exposition, we will break the proof into a number of discrete subsections, providing a tractable roadmap to the final result.

Step 1. $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$ *is nonempty.*

It suffices to show that there exists at least one $\mathscr{Y}$-measurable selection of $\mathcal{X}(Z_Y)$, that is, a $\mathscr{Y}$-measurable random variable, say $X : \Omega \to \mathbb{R}^N$, such that $X(\omega) \in \mathcal{X}(Z_Y(\omega))$, for all ω in the domain of $\mathcal{X}(Z_Y)$.

We first show that the composite multifunction $\mathcal{X}(Z_Y(\cdot)) : \Omega \rightrightarrows \mathbb{R}^N$ is $\mathscr{Y}$-measurable. Recall from Definition 4 that it suffices to show that

$$\mathcal{X} Z_Y^{-1}(\mathcal{A}) \triangleq \left\{\omega \in \Omega \,\middle|\, \mathcal{X}(Z_Y(\omega)) \bigcap \mathcal{A} \neq \varnothing\right\} \in \mathscr{Y}, \tag{8.53}$$

for every closed $\mathcal{A} \subseteq \mathbb{R}^N$. Since the closed-valued multifunction $\mathcal{X}$ is Borel measurable, it is true that $\mathcal{X}^{-1}(\mathcal{A}) \in \mathscr{B}\left(\mathbb{R}^N\right)$, for all closed $\mathcal{A} \subseteq \mathbb{R}^N$. We also know that Z_Y is $\mathscr{Y}$-measurable, or that

$Z_Y^{-1}(\mathcal{B}) \in \mathscr{Y}$, for all $\mathcal{B} \in \mathscr{B}\left(\mathbb{R}^N\right)$. Setting $\mathcal{B} \equiv \mathcal{X}^{-1}(\mathcal{A}) \in \mathscr{B}\left(\mathbb{R}^N\right)$, for any arbitrary closed $\mathcal{A} \subseteq \mathbb{R}^N$, it is true that

$$\begin{aligned}\mathscr{Y} \ni Z_Y^{-1}\left(\mathcal{X}^{-1}(\mathcal{A})\right) &\equiv \left\{\omega \in \Omega \,\middle|\, Z_Y(\omega) \in \mathcal{X}^{-1}(\mathcal{A})\right\} \\ &\equiv \left\{\omega \in \Omega \,\middle|\, \mathcal{X}(Z_Y(\omega)) \bigcap \mathcal{A} \neq \varnothing\right\} \\ &\equiv \mathcal{X} Z_Y^{-1}(\mathcal{A}),\end{aligned} \tag{8.54}$$

and, thus, the composition $\mathcal{X}(Z_Y(\cdot))$ is $\mathscr{Y}$-measurable, or, in other words, measurable on the measurable (sub)space $(\Omega, \mathscr{Y})$.

Now, since the closed-valued multifunction $\mathcal{X}(Z_Y)$ is measurable on $(\Omega, \mathscr{Y})$, it admits a *Castaing Representation* (Theorem 14.5 in [25] & Theorem 7.34 in [26]). Therefore, there exists at least one $\mathscr{Y}$-measurable selection of $\mathcal{X}(Z_Y)$, which means that $\mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$ contains at least one element. ★

Step 2. *ϑ is $\overline{\mathscr{Y}}$-measurable.*

To show the validity of this statement, we first demonstrate that, for any chosen $h : \mathbb{R}^M \times \mathbb{R}^N \to \overline{\mathbb{R}}$, as in Definition 6, the function $\xi : \mathbb{R}^M \to \overline{\mathbb{R}}$, defined as

$$\xi(\boldsymbol{y}) \triangleq \inf_{\boldsymbol{x} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))} h(\boldsymbol{y}, \boldsymbol{x}), \quad \forall \boldsymbol{y} \in \mathbb{R}^M, \tag{8.55}$$

is measurable relative to $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$, the completion of $\mathscr{B}\left(\mathbb{R}^M\right)$ relative to the pushforward $\mathcal{P}_Y$. This follows easily from the following facts. First, the *graph* of the measurable multifunction $\mathcal{X}(\mathcal{Z}(\cdot))$ is itself measurable and in $\mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$ (Theorem 14.8 in [25]), and, therefore, *analytic* (Appendix A.2 in [27]). Second, h is jointly Borel measurable and, therefore, a *lower semianalytic* function (Appendix A.2 in [27]). As a result, ([28], Proposition 7.47) implies that ξ is also lower semianalytic, and, consequently, *universally measurable* (Appendix A.2 in [27]). Being universally measurable, ξ is also measurable relative to $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$, thus proving our claim. We also rely on the definitions of both $\overline{\mathscr{Y}}$ and $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$, stated as (Theorem 1.9 in [48])

$$\mathcal{B} \in \overline{\mathscr{Y}} \iff \mathcal{B} \equiv \mathcal{C} \bigcup \mathcal{D} \,\middle|\, \mathcal{C} \in \mathscr{Y} \text{ and } \mathcal{D} \subseteq \mathcal{O} \in \mathscr{Y}, \text{ with } \mathcal{P}|_{\mathscr{Y}}(\mathcal{O}) \equiv 0 \quad \text{and} \tag{8.56}$$

$$\mathcal{B} \in \overline{\mathscr{B}}\left(\mathbb{R}^M\right) \iff \mathcal{B} \equiv \mathcal{C} \bigcup \mathcal{D} \,\middle|\, \mathcal{C} \in \mathscr{B}\left(\mathbb{R}^M\right) \text{ and } \mathcal{D} \subseteq \mathcal{O} \in \mathscr{B}\left(\mathbb{R}^M\right), \text{ with } \mathcal{P}_Y(\mathcal{O}) \equiv 0. \tag{8.57}$$

Now, specifically, to show that ϑ is measurable relative to $\overline{\mathscr{Y}}$, it suffices to show that, for every Borel $\mathcal{A} \in \mathscr{B}(\overline{\mathbb{R}})$,

$$\vartheta^{-1}(\mathcal{A}) \triangleq \{\omega \in \Omega | \vartheta(\omega) \in \mathcal{A}\} \in \overline{\mathscr{Y}}. \tag{8.58}$$

Recall, that, by definition of ξ, it is true that $\xi(Y(\omega)) \equiv \vartheta(\omega)$, for all $\omega \in \Omega$. Then, for every $\mathcal{A} \in \mathscr{B}(\overline{\mathbb{R}})$, we may write

$$\begin{aligned}\vartheta^{-1}(\mathcal{A}) &\equiv \xi Y^{-1}(\mathcal{A}) \\ &\equiv \{\omega \in \Omega | \xi(Y(\omega)) \in \mathcal{A}\} \\ &\equiv \left\{\omega \in \Omega | Y(\omega) \in \xi^{-1}(\mathcal{A})\right\}\end{aligned}$$

$$\triangleq Y^{-1}\left(\xi^{-1}(\mathcal{A})\right). \tag{8.59}$$

But $\xi^{-1}(\mathcal{A}) \in \overline{\mathscr{B}}\left(\mathbb{R}^M\right)$, which, by (8.57), equivalently means that $\xi^{-1}(\mathcal{A}) \equiv \mathcal{G}_{\mathcal{A}} \bigcup \mathcal{H}_{\mathcal{A}}$, for some $\mathcal{G}_{\mathcal{A}} \in \mathscr{B}\left(\mathbb{R}^M\right)$ and some $\mathcal{H}_{\mathcal{A}} \subseteq \mathcal{E}_{\mathcal{A}} \in \mathscr{B}\left(\mathbb{R}^M\right)$, with $\mathcal{P}_Y(\mathcal{E}_{\mathcal{A}}) \equiv 0$. Thus, we may further express any $\mathcal{A}$-preimage of ϑ as

$$\begin{aligned}\vartheta^{-1}(\mathcal{A}) &\equiv Y^{-1}\left(\mathcal{G}_{\mathcal{A}} \bigcup \mathcal{H}_{\mathcal{A}}\right) \\ &\equiv Y^{-1}(\mathcal{G}_{\mathcal{A}}) \bigcup Y^{-1}(\mathcal{H}_{\mathcal{A}}).\end{aligned} \tag{8.60}$$

Now, because $\mathcal{G}_{\mathcal{A}}$ is Borel and Y is a random element, it is true that $Y^{-1}(\mathcal{G}_{\mathcal{A}}) \in \mathscr{Y}$. On the other hand, $\mathcal{H}_{\mathcal{A}} \subseteq \mathcal{E}_{\mathcal{A}}$, which implies that $Y^{-1}(\mathcal{H}_{\mathcal{A}}) \subseteq Y^{-1}(\mathcal{E}_{\mathcal{A}})$, where

$$\mathcal{P}|_{\mathscr{Y}}\left(Y^{-1}(\mathcal{E}_{\mathcal{A}})\right) \equiv \mathcal{P}_Y(\mathcal{E}_{\mathcal{A}}) \equiv 0. \tag{8.61}$$

Therefore, we have shown that, for every $\mathcal{A} \in \mathscr{B}\left(\overline{\mathbb{R}}\right)$, $\vartheta^{-1}(\mathcal{A})$ may always be written as a union of an element in $\mathscr{Y}$ and some subset of a $\mathcal{P}|_{\mathscr{Y}}$-null set, also in $\mathscr{Y}$. Enough said. ★

Step 3. *For every* $X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, *it is true that* $h(Y,X) \geq \inf_{\boldsymbol{x}\in\mathcal{X}(Z_Y)} h(Y,\boldsymbol{x}) \equiv \vartheta$.

For each $\omega \in \Omega$ (which also determines Y), we may write

$$\begin{aligned}\vartheta(\omega) &\equiv \inf_{\boldsymbol{x}\in\mathcal{X}(\mathcal{Z}(Y(\omega)))} h(Y(\omega),\boldsymbol{x}) \\ &\equiv \inf_{\mathcal{M}(Y(\omega))\in\mathcal{X}(Z_Y(\omega))} h(Y(\omega),\mathcal{M}(Y(\omega))),\end{aligned} \tag{8.62}$$

where $\mathcal{M} : \mathbb{R}^M \to \mathbb{R}^N$ is of arbitrary nature. Therefore, ϑ may be equivalently regarded as the result of infimizing h over the set of all, *measurable or not*, functionals of Y, which are also selections of $\mathcal{X}(Z_Y)$. This set, of course, includes $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$. Now, choose an $X \equiv \mathcal{M}_X(Y) \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, as above, for some Borel measurable $\mathcal{M}_X : \mathbb{R}^M \to \mathbb{R}^N$. Then, it must be true that

$$\vartheta(\omega) \leq h(Y(\omega),\mathcal{M}_X(Y(\omega))) \equiv h(Y(\omega),X(\omega)), \tag{8.63}$$

everywhere in $\omega \in \Omega$. ★

Step 4. *It is also true that*

$$\inf_{X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y,X)\} \geq \mathbb{E}\left\{\inf_{\boldsymbol{x}\in\mathcal{X}(Z_Y)} h(Y,\boldsymbol{x})\right\}. \tag{8.64}$$

From **Step 3**, we know that, for every $X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, we have

$$h(Y,X) \geq \vartheta. \tag{8.65}$$

At this point, we exploit measurability of ϑ, proved in **Step 2**. Since, by assumption,

$$\inf_{X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y,X)\} \equiv \inf_{X\in\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot,X)\} < +\infty, \tag{8.66}$$

it follows that there exists $X_F \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, such that $\mathbb{E}\{h(Y, X_F)\} < +\infty$ (recall that the integral $\mathbb{E}\{g(\cdot, X_F)\}$ exists anyway, also by assumption). Since (8.65) holds for every $X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, it also holds for $X_F \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$ and, consequently, the integral of ϑ exists, with $\mathbb{E}\{\vartheta\} < +\infty$. Then, we may take expectations on both sides of (8.65) (Theorem 1.5.9 (b) in [45]), yielding

$$\mathbb{E}\{h(Y, X)\} \geq \mathbb{E}\{\vartheta\}, \quad \forall X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}. \tag{8.67}$$

Infimizing additionally both sides over $X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, we obtain the desired inequality.

We may also observe that, if $\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y, X)\} \equiv -\infty$, then

$$\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y, X)\} \equiv \mathbb{E}\{\vartheta\} \equiv -\infty, \tag{8.68}$$

and the conclusion of Lemma 3 holds immediately. Therefore, in the following, we may assume that $\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{h(Y, X)\} > -\infty$. ★

Step 5. *For every $\varepsilon > 0$, $n \in \mathbb{N}$ and every $\boldsymbol{y} \in \mathbb{R}^M$, there exists $\boldsymbol{x} \equiv \boldsymbol{x}_{\boldsymbol{y}} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$, such that*

$$h\left(\boldsymbol{y}, \boldsymbol{x}_{\boldsymbol{y}}\right) \leq \max\{\xi(\boldsymbol{y}), -n\} + \varepsilon. \tag{8.69}$$

This simple fact may be shown by contradiction; replacing the universal with existential quantifiers and vice versa in the above statement, suppose that there exists $\varepsilon > 0$, $n \in \mathbb{N}$, and $\boldsymbol{y} \in \mathbb{R}^M$ such that, for all $\boldsymbol{x} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$, $h(\boldsymbol{y}, \boldsymbol{x}) > \max\{\xi(\boldsymbol{y}), -n\} + \varepsilon$. There are two cases: **1)** $\xi(\boldsymbol{y}) > -\infty$. In this case, $\max\{\xi(\boldsymbol{y}), -n\} \geq \xi(\boldsymbol{y})$, which would imply that, for all $\boldsymbol{x} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$,

$$h(\boldsymbol{y}, \boldsymbol{x}) > \xi(\boldsymbol{y}) + \varepsilon, \tag{8.70}$$

contradicting the fact that $\xi(\boldsymbol{y})$ is the infimum (the greatest lower bound) of $h(\boldsymbol{y}, \boldsymbol{x})$ over $\mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$, since $\varepsilon > 0$. **2)** $\xi(\boldsymbol{y}) \equiv -\infty$. Here, $\max\{\xi(\boldsymbol{y}), -n\} \equiv -n$, and, for all $\boldsymbol{x} \in \mathcal{X}_{\mathcal{Z}}(\boldsymbol{y})$, we would write

$$h(\boldsymbol{y}, \boldsymbol{x}) > -n + \varepsilon \in \mathbb{R}, \tag{8.71}$$

which, again, contradicts the fact that $-\infty \equiv \xi(\boldsymbol{y})$ is the infimum of $h(\boldsymbol{y}, \boldsymbol{x})$ over $\mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$. Therefore, in both cases, we are led to a contradiction, implying that the statement preceding and including (8.69) is true. The idea of using the maximum operator, so that $\xi(\boldsymbol{y})$ may be allowed to take the value $-\infty$, is credited to and borrowed from ([25], proof of Theorem 14.60). ★

Step 6. *There exists a Borel measurable function $\widetilde{\xi} : \mathbb{R}^M \to \overline{\mathbb{R}}$, such that*

$$\widetilde{\xi}(\boldsymbol{y}) \equiv \xi(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in \overline{\mathcal{R}}_\xi \supseteq \mathcal{R}_\xi, \tag{8.72}$$

where $\mathcal{R}_\xi \in \mathscr{B}\left(\mathbb{R}^M\right)$ is such that $\mathcal{P}_Y\left(\mathcal{R}_\xi\right) \equiv 1$, and $\overline{\mathcal{R}}_\xi \in \overline{\mathscr{B}}\left(\mathbb{R}^M\right)$ is such that $\overline{\mathcal{P}}_Y\left(\overline{\mathcal{R}}_\xi\right) \equiv 1$, where $\overline{\mathcal{P}}_Y$ denotes the completion of the pushforward $\mathcal{P}_Y$.

From ([48], Proposition 2.12), we know that, since ξ is $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$-measurable, there exists a $\mathscr{B}\left(\mathbb{R}^M\right)$-measurable function $\widetilde{\xi} : \mathbb{R}^M \to \overline{\mathbb{R}}$, such that

$$\widetilde{\xi}(\boldsymbol{y}) \equiv \xi(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in \overline{\mathcal{R}}_\xi, \tag{8.73}$$

where $\overline{\mathcal{R}}_\xi$ is an event in $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$, such that $\overline{\mathcal{P}}_Y\left(\overline{\mathcal{R}}_\xi\right) \equiv 1$. However, from **Step 2** (see (8.57)), we know that $\overline{\mathcal{R}}_\xi \equiv \mathcal{R}_\xi \bigcup \overline{\mathcal{R}}_\xi^E$, where $\mathcal{R}_\xi \in \mathscr{B}\left(\mathbb{R}^M\right)$ and $\overline{\mathcal{P}}_Y\left(\overline{\mathcal{R}}_\xi^E\right) \equiv 0$. Then, it may be easily shown that $\overline{\mathcal{P}}_Y\left(\overline{\mathcal{R}}_\xi\right) \equiv \overline{\mathcal{P}}_Y\left(\mathcal{R}_\xi\right) \equiv 1$ and, since $\overline{\mathcal{P}}_Y$ and $\mathcal{P}_Y$ agree on the elements of $\mathscr{B}\left(\mathbb{R}^M\right)$, $\mathcal{P}_Y\left(\mathcal{R}_\xi\right) \equiv 1$, as well. ★

Step 7. *There exists a $(\mathcal{P}, \varepsilon, n)$-optimal selector $X_n^\varepsilon \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$: For every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exists $X_n^\varepsilon \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, such that*

$$h\left(Y, X_n^\varepsilon\right) \leq \max\left\{\inf_{\boldsymbol{x}\in\mathcal{X}(Z_Y)} h\left(Y, \boldsymbol{x}\right), -n\right\} + \varepsilon, \quad \mathcal{P} - a.e.. \tag{8.74}$$

This is the most crucial property of the problem that needs to be established, in order to reach to the final conclusions of Lemma 3. In this step, we make use of Theorem 9. Because Theorem 9 works on Borel spaces, in the following, it will be necessary to work directly on the state space of the random element Y, equipped with its Borel σ-algebra, and the pushforward $\mathcal{P}_Y$. In the following, we will also make use of the results proved in **Step 5** and **Step 6**.

Recall the definition of ξ in the statement of Lemma 3. We may readily show that the multifunction $\mathcal{X}\left(\mathcal{Z}\left(\cdot\right)\right)$ is $\mathscr{B}\left(\mathbb{R}^M\right)$-measurable. This may be shown in exactly the same way as in **Step 1**, exploitting the hypotheses that the multifunction $\mathcal{X}$ and the function $\mathcal{Z}$ are both Borel measurable. Borel measurability of $\mathcal{X}\left(\mathcal{Z}\left(\cdot\right)\right)$ will be exploited shortly.

Compare the result of **Step 5** with what we would like to prove here; the statement preceding and including (8.69) is not enough for our purposes; what we would actually like is to be able to generate a *selector*, that is, a function of $\boldsymbol{y}$ such that (8.69) would hold at least almost everywhere with respect to $\mathcal{P}_Y$. This is why we need Theorem 9. The idea of using Theorem 9 into this context is credited to and borrowed from [49].

From **Step 2** and **Step 6**, we know that ξ is $\overline{\mathscr{B}}\left(\mathbb{R}^M\right)$-measurable and that there exists a Borel measurable function $\widetilde{\xi} : \mathbb{R}^M \to \overline{\mathbb{R}}$, such that $\widetilde{\xi}\left(\boldsymbol{y}\right) \equiv \xi\left(\boldsymbol{y}\right)$, everywhere in $\boldsymbol{y} \in \mathcal{R}_\xi$, where $\mathcal{R}_\xi \in \mathscr{B}\left(\mathbb{R}^M\right)$ is such that $\mathcal{P}_Y\left(\mathcal{R}_\xi\right) \equiv \overline{\mathcal{P}}_Y\left(\mathcal{R}_\xi\right) \equiv 1$. Then, it follows that

$$\widetilde{\xi}\left(\boldsymbol{y}\right) \equiv \inf_{\boldsymbol{x}\in\mathcal{X}(\mathcal{Z}(\boldsymbol{y}))} h\left(\boldsymbol{y}, \boldsymbol{x}\right), \tag{8.75}$$

for all $\boldsymbol{y} \in \mathcal{R}_\xi$.

Define, for brevity, $\mathcal{X}_{\mathcal{Z}}\left(\boldsymbol{y}\right) \triangleq \mathcal{X}\left(\mathcal{Z}\left(\boldsymbol{y}\right)\right)$, for all $\boldsymbol{y} \in \mathbb{R}^M$. Towards the application of Theorem 9, fix any $\varepsilon > 0$ and any $n \in \mathbb{N}$ and consider the set

$$\Pi_{\mathcal{X}_{\mathcal{Z}}}^{\varepsilon,n} \equiv \left\{ \left(\boldsymbol{y}, \boldsymbol{x}\right) \in \mathbb{R}^M \times \mathbb{R}^N \;\middle|\; \begin{array}{cc} \begin{array}{c} \boldsymbol{x} \in \mathcal{X}\left(\mathcal{Z}\left(\boldsymbol{y}\right)\right) \\ h\left(\boldsymbol{y}, \boldsymbol{x}\right) \leq \max\left\{\widetilde{\xi}\left(\boldsymbol{y}\right), -n\right\} + \varepsilon \end{array}, & \text{if } \boldsymbol{y} \in \mathcal{R}_\xi \\ \boldsymbol{x} \in \mathcal{X}\left(\mathcal{Z}\left(\boldsymbol{y}\right)\right), & \text{if } \boldsymbol{y} \in \mathcal{R}_\xi^c \end{array} \right\}. \tag{8.76}$$

We will show that Π_ε^n constitutes a measurable set in $\mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$. Observe that $\Pi_{\mathcal{X}_{\mathcal{Z}}}^{\varepsilon,n} \equiv \Pi_{\mathcal{X}_{\mathcal{Z}}} \bigcap \left(\Pi^{\varepsilon,n} \bigcup \Pi_{rem}\right)$, where we define

$$\Pi_{\mathcal{X}_{\mathcal{Z}}} \triangleq \left\{ \left(\boldsymbol{y}, \boldsymbol{x}\right) \in \mathbb{R}^M \times \mathbb{R}^N \middle| \boldsymbol{x} \in \mathcal{X}\left(\mathcal{Z}\left(\boldsymbol{y}\right)\right) \right\}, \tag{8.77}$$

$$\Pi^{\varepsilon,n} \triangleq \left\{(\boldsymbol{y},\boldsymbol{x}) \in \mathbb{R}^M \times \mathbb{R}^N \middle| \boldsymbol{y} \in \mathcal{R}_\xi, h(\boldsymbol{y},\boldsymbol{x}) \le \max\left\{\widetilde{\xi}(\boldsymbol{y}), -n\right\} + \varepsilon\right\} \quad \text{and} \tag{8.78}$$

$$\Pi_{rem} \triangleq \left\{(\boldsymbol{y},\boldsymbol{x}) \in \mathbb{R}^M \times \mathbb{R}^N \middle| \boldsymbol{y} \in \mathcal{R}_\xi^c\right\}. \tag{8.79}$$

Clearly, it suffices to show that both $\Pi_{\mathcal{X}_\mathcal{Z}}$ and $\Pi^{\varepsilon,n}$ are in $\mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$. First, the set $\Pi_{\mathcal{X}_\mathcal{Z}}$ is the *graph* of the multifunction $\mathcal{X}_\mathcal{Z}$, and, because $\mathcal{X}_\mathcal{Z}$ is measurable, it follows from ([25], Theorem 14.8) that $\Pi_{\mathcal{X}_\mathcal{Z}} \in \mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$. Second, because g is $\boldsymbol{SP}\Diamond\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, h is jointly Borel measurable. Additionally, $\mathcal{R}_\xi$ is Borel and $\widetilde{\xi}$ is Borel as well. Consequently, $\Pi^{\varepsilon,n}$ can be written as the intersection of two measurable sets, implying that it is in $\mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$, as well. And third, $\Pi_{rem} \in \mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$, since $\mathcal{R}_\xi^c$ is Borel, as a complement of a Borel set. Therefore, $\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}} \in \mathscr{B}\left(\mathbb{R}^M\right) \otimes \mathscr{B}\left(\mathbb{R}^N\right)$.

Now, we have to verify the selection property, set as a requirement in the statement of Theorem 9. Indeed, for every $\boldsymbol{y} \in \mathcal{R}_\xi$, there exists $\boldsymbol{x}_{\boldsymbol{y}} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$, such that (8.69) holds, where $\xi(\boldsymbol{y}) \equiv \widetilde{\xi}(\boldsymbol{y})$ (see **Step 6** and above), while, for every $\boldsymbol{y} \in \mathcal{R}_\xi^c$, any $\boldsymbol{x}_{\boldsymbol{y}} \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y}))$ will do. Thus, for every $\boldsymbol{y} \in \mathbb{R}^M$, there exists $\boldsymbol{x}_{\boldsymbol{y}} \in \mathbb{R}^N$, such that $\left(\boldsymbol{y},\boldsymbol{x}_{\boldsymbol{y}}\right) \in \Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}$. As a result, Theorem 9 applies and implies that there exists a Borel subset $\mathcal{R}^c_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}$ of $\mathcal{P}_Y$-measure 0, as well as a Borel measurable selector $\mathsf{S}^\varepsilon_n : \mathbb{R}^M \to \mathbb{R}^N$, such that, $(\boldsymbol{y}, \mathsf{S}^\varepsilon_n(\boldsymbol{y})) \in \Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}$, for all $\boldsymbol{y} \in \mathcal{R}_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}$. In other words, the Borel measurable selector S^ε_n is such that

$$\begin{gathered} \mathsf{S}^\varepsilon_n(\boldsymbol{y}) \in \mathcal{X}(\mathcal{Z}(\boldsymbol{y})) \quad \text{and} \\ h(\boldsymbol{y}, \mathsf{S}^\varepsilon_n(\boldsymbol{y})) \le \max\left\{\widetilde{\xi}(\boldsymbol{y}), -n\right\} + \varepsilon, \qquad \forall \boldsymbol{y} \in \mathcal{R}_\xi \bigcap \mathcal{R}_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}} \triangleq \mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}, \end{gathered} \tag{8.80}$$

where, of course, $\mathcal{P}_Y\left(\mathcal{R}_\xi \bigcap \mathcal{R}_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right) \equiv 1$. Additionally, (8.80) must be true at $\boldsymbol{y} = Y(\omega)$, as long as ω is such that the values of Y are restricted to $\mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}$. Equivalently, we demand that

$$\omega \in \left\{\omega \in \Omega \middle| Y(\omega) \in \mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right\} \equiv Y^{-1}\left(\mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right) \triangleq \Omega^\varepsilon_n. \tag{8.81}$$

But $\mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}} \in \mathscr{B}\left(\mathbb{R}^M\right)$ and Y is a random element and, hence, a measurable function for Ω to $\mathbb{R}^M$. This means that $\Omega^\varepsilon_n \in \mathscr{Y}$ and we are allowed to write

$$\begin{aligned} \mathcal{P}(\Omega^\varepsilon_n) &\equiv \int_{Y^{-1}\left(\mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right)} \mathcal{P}(\mathrm{d}\omega) \\ &= \int_{\left\{\boldsymbol{y} \in \mathbb{R}^M \middle| \boldsymbol{y} \in \mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right\}} \mathcal{P}_Y(\mathrm{d}\boldsymbol{y}) \\ &\equiv \mathcal{P}_Y\left(\mathcal{R}^\xi_{\Pi^{\varepsilon,n}_{\mathcal{X}_\mathcal{Z}}}\right) \equiv 1. \end{aligned} \tag{8.82}$$

Therefore, we may pull (8.80) back to the base space, and restate it as

$$\begin{gathered} \mathsf{S}^\varepsilon_n(Y(\omega)) \in \mathcal{X}(Z_Y(\omega)) \quad \text{and} \\ h(Y(\omega), \mathsf{S}^\varepsilon_n(Y(\omega))) \le \max\{\xi(Y(\omega)), -n\} + \varepsilon, \qquad \forall \omega \in \Omega^\varepsilon_n, \end{gathered} \tag{8.83}$$

where $\Omega_n^\varepsilon \subseteq \Omega$ is an event, such that $\mathcal{P}(\Omega_n^\varepsilon) \equiv 1$. Then, by construction, $\mathsf{S}_n^\varepsilon(Y) \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$. As a result, for any choice of $\varepsilon > 0$ and $n \in \mathbb{N}$, the selector $X_n^\varepsilon \triangleq \mathsf{S}_n^\varepsilon(Y) \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$ is such that

$$\mathbb{E}\{g(\cdot, X_n^\varepsilon) | \mathscr{Y}\}(\omega) \leq \max\{\vartheta(\omega), -n\} + \varepsilon, \quad \mathcal{P} - a.e.. \tag{8.84}$$

We are done. ★

Step 8. *It is true that*

$$\inf_{X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}} \mathbb{E}\{h(Y, X)\} \leq \mathbb{E}\left\{\inf_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} h(Y, \boldsymbol{x})\right\}. \tag{8.85}$$

Define the sequence of random variables $\{\varpi_n : \Omega \to \overline{\mathbb{R}}\}_{n \in \mathbb{N}}$ as (see the RHS of (8.74))

$$\varpi_n(\omega) \triangleq \max\{\vartheta(\omega), -n\}, \quad \forall (\omega, n) \in \Omega \times \mathbb{N}. \tag{8.86}$$

Also, recall that $\mathbb{E}\{\vartheta\} < +\infty$. Additionally, observe that

$$\varpi_n(\omega) \leq \max\{\vartheta(\omega), 0\} \geq 0, \quad \forall (\omega, n) \in \Omega \times \mathbb{N}, \tag{8.87}$$

where it is easy to show that $\mathbb{E}\{\max\{\vartheta, 0\}\} < +\infty$. Thus, all members of $\{\varpi_n\}_{n \in \mathbb{N}}$ are bounded by an integrable random variable, everywhere in ω and uniformly in n, whereas it is trivial that, for every $\omega \in \Omega$, $\varpi_n(\omega) \underset{n \to \infty}{\searrow} \vartheta(\omega)$.

Consider now the result of **Step 7**, where we showed that, for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exists a selector $X_n^\varepsilon \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, such that

$$h(Y, X_n^\varepsilon) \leq \varpi_n + \varepsilon, \quad \mathcal{P} - a.e.. \tag{8.88}$$

We can then take expectations on both sides (note that all involved integrals exist), to obtain

$$\mathbb{E}\{h(Y, X_n^\varepsilon)\} \leq \mathbb{E}\{\varpi_n\} + \varepsilon. \tag{8.89}$$

Since $X_n^\varepsilon \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, it also follows that

$$\inf_{X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}} \mathbb{E}\{h(Y, X)\} \leq \mathbb{E}\{\varpi_n\} + \varepsilon. \tag{8.90}$$

It is also easy to see that ϖ_n fulfills the requirements of the Extended Monotone Convergence Theorem ([45], Theorem 1.6.7 (b)). Therefore, we may pass to the limit on both sides of (8.90) as $n \to \infty$, yielding

$$\inf_{X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}} \mathbb{E}\{h(Y, X)\} \leq \mathbb{E}\{\vartheta\} + \varepsilon. \tag{8.91}$$

But $\varepsilon > 0$ is arbitrary. ★

Finally, just combine the statements of **Step 4** and **Step 8**, and the result follows, completing the proof of Lemma 3. ■

Remark 15. Obviously, Lemma 3 holds also for maximization problems as well, by defining $g \equiv -f$, for some random function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$, under the corresponding setting and assumptions. Note that, in this case, we have to assume that $\sup_{X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}} \mathbb{E}\{f(\cdot, X)\} > -\infty$. ■

Remark 16. Lemma 3 may be considered a useful variation of Theorem 14.60 in [25], in the following sense. First, it is specialized for conditional expectations of random functions, which are additionally $\boldsymbol{SP}\Diamond\mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, in the context of stochastic control. The latter property allows these conditional expectations to be expressed as (Borel) random functions themselves. This is in contrast to ([25], Theorem 14.60), where it is assumed that the random function, whose role is played by the respective conditional expectation in Lemma 3, is somehow provided apriori. Second, Lemma 3 extends ([25], Theorem 14.60), in the sense that the decision set $\mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$ confines any solution to the respective optimization problem to be a $\mathscr{Y}$-measurable selection of a closed-valued measurable multifunction, while at the same time, apart from natural (and important) measurability requirements, no continuity assumptions are imposed on the structure of the random function induced by the respective conditional expectation; only the validity of the substitution property is required. In ([25], Theorem 14.60), on the other hand, it is respectively assumed that the involved random function is a normal integrand, or, in other words, that it is random lower semicontinuous. ■

Remark 17. In Lemma 3, variational optimization is performed over some subset of functions *measurable relative to* $\mathscr{Y} \equiv \sigma\{Y\}$, where Y is some given random element. Although we do not pursue such an approach here, it would most probably be possible to develop a more general version of Lemma 3, where the decision set would be appropriately extended to include $\overline{\mathscr{Y}}$-measurable random elements, as well. In such case, the definition of the substitution property could be extended under the framework of lower semianalytic functions and universal measurability, and would allow the development of arguments showing existence of everywhere ε-optimal and potentially everywhere optimal policies (decisions), in the spirit of [27, 28]. ■

8.2.3 Guaranteeing the Existence of Measurable Optimal Controls

Although Lemma 3 constitutes a very useful result, which enables the simplification of a stochastic variational problem, by essentially replacing it by an at least structurally simpler, pointwise optimization problem, it does not provide insight on the existence of a common optimal solution, *within the respective decision sets.*

On the one hand, it is easy to observe that, similarly to ([25], Theorem 14.60), if there exists an optimal selection $X^* \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, such that

$$X^* \in \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathcal{X}(Z_Y)} \mathbb{E}\left\{ g\left(\cdot,\boldsymbol{x}\right) \middle| \mathscr{Y}\right\} \neq \varnothing, \quad \mathcal{P}-a.e., \tag{8.92}$$

and Lemma 3 applies, then, exploitting the fact that g is $\boldsymbol{SP}\Diamond\mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$, we may write

$$\begin{aligned}\inf_{X\in\mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}} \mathbb{E}\left\{g\left(\cdot,X\right)\right\} &\equiv \mathbb{E}\left\{\vartheta\right\} \equiv \mathbb{E}\left\{\xi\left(Y\right)\right\} \\ &= \mathbb{E}\left\{h\left(Y,X^*\right)\right\} \\ &= \mathbb{E}\left\{\mathbb{E}\left\{g\left(\cdot,X^*\right)\middle|\mathscr{Y}\right\}\right\} \\ &\equiv \mathbb{E}\left\{g\left(\cdot,X^*\right)\right\},\end{aligned} \tag{8.93}$$

implying that the infimum of $\mathbb{E}\{g(\cdot,X)\}$ over $X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$ is attained by X^*; therefore, X^* is also an optimal solution to the respective variational problem. *Conversely*, if X^* attains the infimum of $\mathbb{E}\{g(\cdot,X)\}$ over $X \in \mathcal{F}_{\mathcal{X}(Z_Y)}^{\mathscr{Y}}$ *and* the infimum is greater than $-\infty$, then both $\mathbb{E}\left\{g\left(\cdot,X^*\right)\right\}$ and

$\mathbb{E}\{\vartheta\}$ are finite, which also implies that $\mathbb{E}\left\{ g\left(\cdot, X^*\right) \middle| \mathscr{Y} \right\}$ and ϑ are finite $\mathcal{P}-a.e.$. As a result, and recalling **Step 3** in the proof of Lemma 3, we have

$$\mathbb{E}\left\{\mathbb{E}\left\{ g\left(\cdot, X^*\right) \middle| \mathscr{Y} \right\} - \vartheta\right\} \equiv 0 \quad \text{and} \tag{8.94}$$

$$\mathbb{E}\left\{ g\left(\cdot, X^*\right) \middle| \mathscr{Y} \right\} - \vartheta \geq 0, \quad \mathcal{P}-a.e.. \tag{8.95}$$

and, consequently, $\vartheta \equiv \mathbb{E}\left\{ g\left(\cdot, X^*\right) \middle| \mathscr{Y} \right\}$, $\mathcal{P}-a.e.$.

Unfortunately, it is not possible to guarantee existence of such an $X^* \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, in general. However, at least for the purposes of this paper, it is both reasonable and desirable to demand the existence of an optimal solution X^*, satisfying (8.92) (in our spatially controlled beamforming problem, we *need* to make a *feasible* decision on the position of the relays at the next time slot). Additionally, such an optimal solution, if it exists, will not be available in closed form, and, consequently, it will be impossible to verify measurability directly. Therefore, we have to be able to show both existence and measurability of X^* *indirectly*, and specifically, by imposing constraints on the structure of the stochastic optimization problem under consideration. One way to do this, *emphasizing on our spatially controlled beamforming problem formulation*, is to restrict our attention to pointwise optimization problems involving Carathéodory objectives, over closed-valued multifunctions, which are additionally *closed* -see Definition 5.

Focusing on Carathéodory functions is not particularly restrictive, since it is already clear that, in order to guarantee the validity of the substitution rule (the ***SP*** Property), similar continuity assumptions would have to be imposed on both random functions g and $\mathbb{E}\{ g(\cdot,\cdot)| \mathscr{Y}\} \equiv h$, as Theorem 8 suggests. At the same time, restricting our attention to optimizing Carathéodory functions over measurable multifunctions, measurability of optimal values and optimal decisions is preserved, as the next theorem suggests.

Theorem 10. (Measurability under Partial Minimization) *On the base subspace* $(\Omega, \mathscr{Y}, \mathcal{P}|_{\mathscr{Y}})$, *where* $\mathscr{Y} \subseteq \mathscr{F}$, *let the random function* $H : \Omega \times \mathbb{R}^N \to \overline{\mathbb{R}}$ *be Carathéodory, and consider another random element* $Z : \Omega \to \mathbb{R}^N$, *as well as any compact-valued multifunction* $\mathcal{X} : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$, *with* $\mathrm{dom}(\mathcal{X}) \equiv \mathbb{R}^N$, *which is also* ***closed****. Additionally, define* $H^* : \Omega \to \overline{\mathbb{R}}$ *as the optimal value to the optimization problem*

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & H(\omega, \boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}(Z(\omega)) \end{array}, \quad \forall \omega \in \Omega. \tag{8.96}$$

Then, H^* *is* $\mathscr{Y}$*-measurable and attained for at least one* $\mathscr{Y}$*-measurable minimizer* $X^* : \Omega \to \mathbb{R}^N$*. If the minimizer* X^* *is unique, then it has to be* $\mathscr{Y}$*-measurable.*

Proof of Theorem 10. From ([26], pp. 365 - 367 and/or [25], Example 14.32 & Theorem 14.37), we may immediately deduce that H^* is $\mathscr{Y}$-measurable and attained for at least one $\mathscr{Y}$-measurable minimizer X^*, as long as the compact (therefore closed, as well)-valued multifunction $\mathcal{X}(Z(\cdot)) : \Omega \rightrightarrows \mathbb{R}^N$ is measurable relative to $\mathscr{Y}$. In order to show that the composition $\mathcal{X}(Z(\cdot))$ is $\mathscr{Y}$-measurable, we use the assumption that the compact-valued multifunction $\mathcal{X} : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is closed and, therefore, Borel measurable (Remark 28 in [26], p. 365). Then, $\mathscr{Y}$-measurability of $\mathcal{X}(Z(\cdot))$ follows by the same arguments as in **Step 1**, in the proof of Lemma 3. ■

Remark 18. It would be important to mention that if one replaces $\mathbb{R}^N$ with any compact (say) subset $\mathcal{H} \subset \mathbb{R}^N$ in the statement of Theorem 10, then the result continues to hold as is. No modification is necessary. In our spatially controlled beamforming problem, this compact set $\mathcal{H}$ is specifically identified either with the hypercubic region $\mathcal{S}^R$, or with some compact subset of it. ■

8.2.4 Fusion & Derivation of Conditions C1-C6

Finally, combining Theorem 8, Lemma 3 and Theorem 10, we may directly formulate the following constrained version of the Fundamental Lemma, which is of central importance regarding the special class of stochastic problems considered in this work and, in particular, (4.5).

Lemma 4. (Fundamental Lemma / Fused Version) *On $(\Omega, \mathscr{F}, \mathcal{P})$, consider a random element $Y : \Omega \to \mathbb{R}^M$, the sub σ-algebra $\mathscr{Y} \triangleq \sigma\{Y\} \subseteq \mathscr{F}$, a random function $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, such that $\mathbb{E}\{g(\cdot, \boldsymbol{x})\}$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$, a multifunction $\mathcal{X} : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$, with $\mathrm{dom}(\mathcal{X}) \equiv \mathbb{R}^N$ and , as well as another function $Z_Y : \Omega \to \mathbb{R}^N$. Assume that:*

C1. *$\mathcal{X}$ is compact-valued and closed, and that*

C2. *Z_Y is a $\mathscr{Y}$-measurable random element.*

Consider also the nonempty decision set $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$. Additionally, suppose that:

C3. *$\mathbb{E}\{g(\cdot, X)\}$ exists for all $X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, with $\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot, X)\} < +\infty$,*

C4. *g is dominated by a $\mathcal{P}$-integrable function, uniformly in $\boldsymbol{x} \in \mathbb{R}^N$,*

C5. *g is Carathéodory on $\Omega \times \mathbb{R}^N$, and that*

C6. *$\mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\} \equiv h(Y, \boldsymbol{x})$ is Carathéodory on $\Omega \times \mathbb{R}^N$.*

Then, the optimal value function $\inf_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} \mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\} \triangleq \vartheta$ is $\mathscr{Y}$-measurable, and it is true that

$$\inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot, X)\} \equiv \mathbb{E}\{\vartheta\} \equiv \mathbb{E}\left\{g\left(\cdot, X^*\right)\right\}, \tag{8.97}$$

for at least one $X^* \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, such that $X^*(\omega) \in \arg\min_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} \mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\}(\omega)$, everywhere in $\omega \in \Omega$. If there is only one minimizer attaining ϑ, then it has to be $\mathscr{Y}$-measurable.

Proof of Lemma 4. We just carefully combine Theorem 8, Lemma 3 and Theorem 10. First, if conditions **C4-C6** are satisfied, then, from Theorem 8, it follows that g is $\boldsymbol{SP}\Diamond\mathfrak{I}_{\mathscr{Y}}$. Then, since $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)} \subseteq \mathfrak{I}_{\mathscr{Y}}$, g is $\boldsymbol{SP}\Diamond\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, as well. Consequently, with **C1-C3** being true, all assumptions of Lemma 3 are satisfied, and the first equivalence of (8.97) from the left is true. Additionally, from Theorem 10, it easily follows that the optimal value ϑ is $\mathscr{Y}$-measurable, attained by an at least one $\mathscr{Y}$-measurable X^*, which, of course, constitutes a selection of $\mathcal{X}(\mathcal{Z}(Y)) \equiv \mathcal{X}(Z_Y)$, or, equivalently, $X^* \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$. Then, because g is $\boldsymbol{SP}\Diamond\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, we may write

$$\begin{aligned}
\vartheta &\equiv \inf_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} \mathbb{E}\{g(\cdot, \boldsymbol{x}) | \mathscr{Y}\} \\
&\equiv \inf_{\boldsymbol{x} \in \mathcal{X}(Z_Y)} h(Y, \boldsymbol{x}) \\
&\equiv h\left(Y, X^*\right) \\
&\equiv \mathbb{E}\left\{g\left(\cdot, X^*\right) \middle| \mathscr{Y}\right\}, \quad \mathcal{P}-a.e.,
\end{aligned} \tag{8.98}$$

which yields the equivalence $\mathbb{E}\{\vartheta\} \equiv \mathbb{E}\left\{g\left(\cdot, X^*\right)\right\}$. The proof is complete. ■

Remark 19. Note that, because, in Lemma 4, $X^*(\omega) \in \mathcal{X}(Z_Y(\omega))$, *everywhere* in $\omega \in \Omega$, it is true that X^* is actually a minimizer of the slightly more constrained problem of infimizing $\mathbb{E}\{g(\cdot, X)\}$ over the set of *precisely* all $\mathscr{Y}$-measurable selections of $\mathcal{X}(Z_Y)$. Denoting this decision set as $\mathcal{F}^{\mathscr{Y},E}_{\mathcal{X}(Z_Y)} \subseteq \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$, the aforementioned statement is true since, simply,

$$\inf_{X \in \mathcal{F}^{\mathscr{Y},E}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot, X)\} \geq \inf_{X \in \mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot, X)\} \equiv \mathbb{E}\left\{g\left(\cdot, X^*\right)\right\} \tag{8.99}$$

$$\Longrightarrow \inf_{X \in \mathcal{F}^{\mathscr{Y},E}_{\mathcal{X}(Z_Y)}} \mathbb{E}\{g(\cdot, X)\} \equiv \mathbb{E}\left\{g\left(\cdot, X^*\right)\right\}. \tag{8.100}$$

where we have used the fact that $X^* \in \mathcal{F}^{\mathscr{Y},E}_{\mathcal{X}(Z_Y)}$. This type of decision set is considered, for simplicity, in (4.5), which corresponds to the original formulation of the spatially controlled beamforming problem. ■

Lemma 4 is of major importance, as it directly provides us with conditions **C1-C6**, which, being relatively easily verifiable, at least for our spatially controlled beamforming setting, ensure strict theoretical consistency of the methods developed in this paper. At this point, our discussion concerning the Fundamental Lemma has been concluded. ■

8.3 Appendix C: Proofs / Section 4

8.3.1 Proof of Theorem 3

Since, in the following, we are going to verify conditions **C1-C6** of Lemma 4 in Section 8.2.4 (Appendix B) for the 2-stage problem (4.15), it will be useful to first match it to the setting of Lemma 4, term-by-term. Table 1 shows how the components of (4.15) are matched to the respective components of the optimization problem considered in Lemma 4. For the rest of the proof, we consider this variable matching automatic.

Keep $t \in \mathbb{N}^2_{N_T}$ *fixed*. As in the statement of Theorem 3, suppose that, at time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, $\mathbf{p}^o(t-1) \equiv \mathbf{p}^o(\omega, t-1)$ is measurable relative to $\mathscr{C}(\mathcal{T}_{t-2})$. Then, condition **C2** is automatically verified. ★

Next, let us verify **C1**. For this, we will simply show directly that closed-valued translated multifunctions, in the sense of Definition 1, are also closed. Given two *closed* sets $\mathcal{H} \subset \mathbb{R}^N$, $\mathcal{A} \subseteq \mathbb{R}^N$ and a fixed reference $\boldsymbol{h} \in \mathcal{H}$, let $\mathcal{D} : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ be $(\mathcal{H}, \boldsymbol{h})$-translated in $\mathcal{A}$ and consider any two arbitrary sequences

$$\{\boldsymbol{x}_k \in \mathcal{A}\}_{k \in \mathbb{N}} \quad \text{and} \quad \{\boldsymbol{y}_k \in \mathcal{A} - \boldsymbol{h}\}_{k \in \mathbb{N}}, \tag{8.101}$$

such that $\boldsymbol{x}_k \underset{k\to\infty}{\longrightarrow} \boldsymbol{x}$, $\boldsymbol{y}_k \underset{k\to\infty}{\longrightarrow} \boldsymbol{y}$ and $\boldsymbol{x}_k \in \mathcal{D}(\boldsymbol{y}_k)$, for all $k \in \mathbb{N}$. By Definition 1, $\boldsymbol{x}_k \in \mathcal{D}(\boldsymbol{y}_k)$ if and only if $\boldsymbol{x}_k - \boldsymbol{y}_k \in \mathcal{H}$, for all $k \in \mathbb{N}$. But $\boldsymbol{x}_k - \boldsymbol{y}_k \underset{k\to\infty}{\longrightarrow} \boldsymbol{x} - \boldsymbol{y}$ and $\mathcal{H}$ is closed. Therefore, it is true that $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{H}$, as well, showing that $\mathcal{D}$ is closed. By Assumption 2, $\mathcal{C} : \mathbb{R}^{2R} \rightrightarrows \mathbb{R}^{2R}$ is the $(\mathcal{G}, \mathbf{0})$-translated multifunction in $\mathcal{S}^R$, for some compact and, hence, closed, $\mathcal{G} \subset \mathcal{S}^R$. Consequently, the restriction of $\mathcal{C}$ in $\mathcal{S}^R$ is closed and **C3** is verified. ★

Condition **C5** is also easily verified; it suffices to show that both functions $|f(\cdot, \cdot, t)|^2$ and $|g(\cdot, \cdot, t)|^2$ are Carathéodory on $\Omega \times \mathcal{S}$, or, in other words, that the fields $|f(\mathbf{p}, t)|^2$ and $|g(\mathbf{p}, t)|^2$ are everywhere sample path continuous. Indeed, if this holds, $V_I(\cdot, \cdot, t)$ will be Carathéodory, as a

Problem of Lemma 4	2-Stage Problem (4.15)
Random element $Y:\Omega\rightarrow\mathbb{R}^M$	*All* relay positions *and* channel observations, up to (current) time slot $t-1$
σ-Algebra $\mathscr{Y}\triangleq\sigma\{Y\}$	σ-Algebra $\mathscr{C}(\mathcal{T}_{t-1})$, jointly generated by the above random vector
Random Function $g:\Omega\times\mathbb{R}^N\rightarrow\mathbb{R}$	Optimal value of the second-stage problem, $V(\cdot,\cdot,t-1):\Omega\times\mathcal{S}^R\rightarrow\mathbb{R}_{++}$
Multifunction $\mathcal{X}:\mathbb{R}^N\rightrightarrows\mathbb{R}^N$, with $\mathrm{dom}(\mathcal{X})\equiv\mathbb{R}^N$	Spatially feasible motion region $\mathcal{C}:\mathcal{S}^R\rightrightarrows\mathcal{S}^R$, with $\mathrm{dom}(\mathcal{C})\equiv\mathcal{S}^R$
Function $Z_Y:\Omega\rightarrow\mathbb{R}^N$	*Selected* motion policy at time slot $t-2$, $\mathbf{p}^o(\cdot,t-1):\Omega\rightarrow\mathcal{S}^R$
Decision set $\mathcal{F}^{\mathscr{Y}}_{\mathcal{X}(Z_Y)}$	Decision set $\mathcal{D}_t$ (precisely matched with $\mathcal{F}^{\mathscr{Y},E}_{\mathcal{X}(Z_Y)}$)

Table 1: Variable matching for (4.15) and the respective problem considered in Lemma 4.

continuous functional of $|f(\cdot,\cdot,t)|^2$ and $|g(\cdot,\cdot,t)|^2$, and since

$$V\left(\left[\mathbf{p}_1^{\boldsymbol{T}}\ \ldots\ \mathbf{p}_R^{\boldsymbol{T}}\right]^{\boldsymbol{T}},t\right)\equiv\sum_{i\in\mathbb{N}_R^+}V_I(\mathbf{p}_i,t),\tag{8.102}$$

it readily follows that $V(\cdot,\cdot,t)$ is Carathéodory on $\Omega\times\mathcal{S}^R$. In order to show (everywhere) sample path continuity of $|f(\mathbf{p},t)|^2$ (respectively $|g(\mathbf{p},t)|^2$) on $\mathcal{S}$, we may utilize (3.11). As a result, sample path continuity of $|f(\mathbf{p},t)|^2$ is equivalent to sample path continuity of

$$F(\mathbf{p},t)\equiv\alpha_S(\mathbf{p})\,\ell+\sigma_S(\mathbf{p},t)+\xi_S(\mathbf{p},t),\quad\forall\mathbf{p}\in\mathcal{S}.\tag{8.103}$$

Of course, α_S is a continuous function of $\mathbf{p}$. As long as the fields $\sigma_S(\mathbf{p},t)$ and $\xi_S(\mathbf{p},t)$ are concerned, these are also sample path continuous; see Section 3.3. Enough said. ★

We continue with **C3**. Since we already know that $V(\cdot,\cdot,t)$ is Carathéodory, it follows from ([44], Lemma 4.51) that $V(\cdot,\cdot,t)$ is also jointly measurable relative to $\mathscr{F}\otimes\mathscr{B}\left(\mathcal{S}^R\right)$. Next, let $\mathbf{p}(t)\equiv\mathbf{p}(\omega,t)\in\mathcal{S}^R$ be *any* random element, measurable with respect to $\mathscr{C}(\mathcal{T}_{t-1})$ and, thus, $\mathscr{F}$, too. Then, from ([44], Lemma 4.49), we know that the pair $(\mathbf{p}(t,\omega),(t,\omega))$ is also $\mathscr{F}$-measurable. Consequently, $|V(\cdot,\mathbf{p}(\cdot,t),t)|^2$ must be $\mathscr{F}$-measurable, as a composition of measurable functions. Additionally, $V(\cdot,\cdot,t)$ is, by definition, nonnegative. Thus, its expectation exists (Corollary 1.6.4 in [45]), and we are done. ★

Conditions **C4** and **C6** need slightly more work, in order to be established. To verify **C4**, we have to show existence of a function in $\mathcal{L}_1(\Omega,\mathscr{F},\mathcal{P};\mathbb{R})$, which dominates $V(\cdot,\cdot,t)$, *uniformly* in $\mathbf{p}\in\mathcal{S}^R$. Everywhere in Ω, again using (3.11), and with $\varsigma\triangleq\log(10)/10$ for brevity, we may write

$$\begin{aligned}V\left(\left[\mathbf{p}_1^{\boldsymbol{T}}\ \ldots\ \mathbf{p}_R^{\boldsymbol{T}}\right]^{\boldsymbol{T}},t\right)&\equiv\sum_{i\in\mathbb{N}_R^+}\frac{P_cP_0\,|f(\mathbf{p}_i,t)|^2\,|g(\mathbf{p}_i,t)|^2}{P_0\sigma_D^2\,|f(\mathbf{p}_i,t)|^2+P_c\sigma^2\,|g(\mathbf{p}_i,t)|^2+\sigma^2\sigma_D^2}\\&\leq\frac{P_0}{\sigma^2}\sum_{i\in\mathbb{N}_R^+}|f(\mathbf{p}_i,t)|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{P_0}{\sigma^2} \sum_{i \in \mathbb{N}_R^+} \sup_{\mathbf{p}_i \in \mathcal{S}} \left| f\left(\mathbf{p}_i, t\right)\right|^2 \\
&\equiv \frac{10^{\rho/10} P_0 R}{\sigma^2} \sup_{\mathbf{p} \in \mathcal{S}} \exp\left(\varsigma F\left(\mathbf{p}, t\right)\right) \\
&\equiv \frac{10^{\rho/10} P_0 R}{\sigma^2} \exp\left(\varsigma \sup_{\mathbf{p} \in \mathcal{S}} F\left(\mathbf{p}, t\right)\right) \\
&\equiv \frac{10^{\rho/10} P_0 R}{\sigma^2} \exp\left(\varsigma \sup_{\mathbf{p} \in \mathcal{S}} \alpha_S\left(\mathbf{p}\right) \ell + \sigma_S\left(\mathbf{p}, t\right) + \xi_S\left(\mathbf{p}, t\right)\right) \\
&\triangleq \frac{10^{\rho/10} P_0 R}{\sigma^2} \exp\left(\varsigma \sup_{\mathbf{p} \in \mathcal{S}} \alpha_S\left(\mathbf{p}\right) \ell + \chi_S\left(\mathbf{p}, t\right)\right) \\
&\leq \frac{10^{\rho/10} P_0 R}{\sigma^2} \exp\left(\varsigma \ell \sup_{\mathbf{p} \in \mathcal{S}} \alpha_S\left(\mathbf{p}\right)\right) \exp\left(\varsigma \sup_{\mathbf{p} \in \mathcal{S}} \chi_S\left(\mathbf{p}, t\right)\right) \\
&\triangleq \varphi\left(\omega, t\right) > 0, \quad \forall \omega \in \Omega.
\end{aligned} \tag{8.104}
$$

Due to the fact that α_S is continuous in $\mathbf{p} \in \mathcal{S}$ and that $\mathcal{S}$ is compact, the Extreme Value Theorem implies that the deterministic term $\sup_{\mathbf{p} \in \mathcal{S}} \alpha_S\left(\mathbf{p}\right)$ is finite. Consequently, it suffices to show that

$$
\mathbb{E}\left\{\exp\left(\varsigma \sup_{\mathbf{p} \in \mathcal{S}} \chi_S\left(\mathbf{p}, t\right)\right)\right\} < +\infty, \tag{8.105}
$$

provided, of course, that the expectation is meaningfully defined. For this to happen, it suffices that the function $\sup_{\mathbf{p} \in \mathcal{S}} \chi_S\left(\mathbf{p}, t\right)$ is a well defined random variable. Since both $\sigma_S\left(\mathbf{p}, t\right)$ and $\xi_S\left(\mathbf{p}, t\right)$ are sample path continuous, it follows that the sum field $\sigma_S\left(\mathbf{p}, t\right) + \xi_S\left(\mathbf{p}, t\right)$ is sample path continuous. It is then relatively easy to see that $\sup_{\mathbf{p} \in \mathcal{S}} \chi_S\left(\mathbf{p}, t\right)$ is a measurable function. See, for instance, Theorem 10, or [38]. Additionally, the Extreme Value Theorem again implies that $\sup_{\mathbf{p} \in \mathcal{S}} \chi_S\left(\mathbf{p}, t\right)$ is finite *everywhere* on Ω, which in turn means that the field $\chi_S\left(\mathbf{p}, t\right)$ is *at least* almost everywhere bounded on the compact set $\mathcal{S}$.

Now, in order to prove that (8.105) is indeed true, we will invoke a well known result from the theory of concentration of measure, the *Borell-TIS Inequality*, which now follows.

Theorem 11. (Borell-TIS Inequality [38]) *Let $X\left(\boldsymbol{s}\right)$, $\boldsymbol{s} \in \mathbb{R}^N$, be a real-valued, zero-mean, Gaussian random field, $\mathcal{P}$-almost everywhere bounded on a compact subset $\mathcal{K} \subset \mathbb{R}^N$. Then, it is true that*

$$
\mathbb{E}\left\{\sup_{\boldsymbol{s} \in \mathcal{K}} X\left(\boldsymbol{s}\right)\right\} < +\infty \quad \textit{and} \tag{8.106}
$$

$$
\mathcal{P}\left(\sup_{\boldsymbol{s} \in \mathcal{K}} X\left(\boldsymbol{s}\right) - \mathbb{E}\left\{\sup_{\boldsymbol{s} \in \mathcal{K}} X\left(\boldsymbol{s}\right)\right\} > u\right) \leq \exp\left(-\frac{u^2}{2 \sup_{\boldsymbol{s} \in \mathcal{K}} \mathbb{E}\left\{X^2\left(\boldsymbol{s}\right)\right\}}\right), \tag{8.107}
$$

for all $u > 0$.

As highlighted in ([38], page 50), an immediate consequence of the Borell-TIS Inequality is that, under the setting of Theorem 11, we may further assert that

$$\mathcal{P}\left(\sup_{\boldsymbol{s}\in\mathcal{K}} X\left(\boldsymbol{s}\right) > u\right) \leq \exp\left(-\frac{\left(u - \mathbb{E}\left\{\sup_{\boldsymbol{s}\in\mathcal{K}} X\left(\boldsymbol{s}\right)\right\}\right)^2}{2\sup_{\boldsymbol{s}\in\mathcal{K}} \mathbb{E}\left\{X^2\left(\boldsymbol{s}\right)\right\}}\right), \tag{8.108}$$

for all $u > \mathbb{E}\left\{\sup_{\boldsymbol{s}\in\mathcal{K}} X\left(\boldsymbol{s}\right)\right\}$.

To show (8.105), we exploit the Borell-TIS Inequality and follow a procedure similar to ([38], Theorem 2.1.2). First, from the discussion above, we readily see that the field $\chi_S\left(\mathbf{p},t\right)$ does satisfy the assumptions Theorem 11. Also, because $\chi_S\left(\mathbf{p},t\right)$ is the sum of two independent fields, it is true that

$$\mathbb{E}\left\{\chi_S^2\left(\mathbf{p},t\right)\right\} \equiv \eta^2 + \sigma_\xi^2. \tag{8.109}$$

As a result, Theorem 11 implies that $\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right\}$ is finite and we may safely write

$$\begin{aligned}\mathbb{E}\left\{\exp\left(\varsigma\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right)\right\} &\equiv \int_0^\infty \mathcal{P}\left(\exp\left(\varsigma\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right) > x\right)\mathrm{d}x \\ &\equiv \int_0^\infty \mathcal{P}\left(\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right) > \frac{\log\left(x\right)}{\varsigma}\right)\mathrm{d}x.\end{aligned} \tag{8.110}$$

In order to exploit (8.108), it must hold that

$$\frac{\log\left(x\right)}{\varsigma} > \mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right\} \Leftrightarrow x > \exp\left(\varsigma\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right\}\right) > 0. \tag{8.111}$$

Therefore, we may break (8.110) into two parts and bound from above, namely,

$$\begin{aligned}&\mathbb{E}\left\{\exp\left(\varsigma\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right)\right\} \\ &\quad\equiv \int_0^{\exp\left(\varsigma\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)} \mathcal{P}\left(\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right) > \frac{\log\left(x\right)}{\varsigma}\right)\mathrm{d}x \\ &\quad\quad + \int_{\exp\left(\varsigma\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)}^\infty \mathcal{P}\left(\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right) > \frac{\log\left(x\right)}{\varsigma}\right)\mathrm{d}x \\ &\quad\leq \int_0^{\exp\left(\varsigma\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)} \mathrm{d}x \\ &\quad\quad + \int_{\exp\left(\varsigma\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)}^\infty \exp\left(-\frac{\left(\frac{\log\left(x\right)}{\varsigma} - \mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S\left(\mathbf{p},t\right)\right\}\right)^2}{2\left(\eta^2 + \sigma_\xi^2\right)}\right)\mathrm{d}x\end{aligned}$$

$$
\leq \exp\left(\varsigma \mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)
$$

$$
+\varsigma \int_{\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}}^{\infty} \exp(\varsigma u) \exp\left(-\frac{\left(u-\mathbb{E}\left\{\sup_{\mathbf{p}\in\mathcal{S}} \chi_S(\mathbf{p},t)\right\}\right)^2}{2\left(\eta^2+\sigma_\xi^2\right)}\right) \mathrm{d}u. \tag{8.112}
$$

Since both terms on the RHS of (8.112) are finite, (8.105) is indeed satisfied. Consequently, it is true that

$$
\mathbb{E}\{\varphi(\cdot,t)\} < +\infty \Leftrightarrow \varphi(\cdot,t) \in \mathcal{L}_1(\Omega,\mathscr{F},\mathcal{P};\mathbb{R}). \tag{8.113}
$$

Enough said; **C4** is now verified. ★

Moving on to **C6**, the goal here is to show that, for each fixed $t \in \mathbb{N}^2_{N_T}$, the well defined random function $H : \Omega \times \mathcal{S}^R \to \overline{\mathbb{R}}$, defined as

$$
H(\omega,\mathbf{p}) \triangleq \mathbb{E}\{V(\mathbf{p},t)\,|\mathscr{C}(\mathcal{T}_{t-1})\}(\omega), \tag{8.114}
$$

is Carathéodory. Observe, though, that we may write

$$
H(\omega,\mathbf{p}) \equiv \sum_{i\in\mathbb{N}_R^+} H_I(\omega,\mathbf{p}_i), \tag{8.115}
$$

where the random function $H_I : \Omega \times \mathcal{S} \to \mathbb{R}$ is defined as

$$
H_I(\omega,\mathbf{p}) \triangleq \mathbb{E}\left\{\left.\frac{P_c P_0 |f(\mathbf{p},t)|^2 |g(\mathbf{p},t)|^2}{P_0\sigma_D^2 |f(\mathbf{p},t)|^2 + P_c\sigma^2 |g(\mathbf{p},t)|^2 + \sigma^2\sigma_D^2}\right| \mathscr{C}(\mathcal{T}_{t-1})\right\}(\omega). \tag{8.116}
$$

Because a finite sum of Carathéodory functions (in this case, in different variables) is obviously Carathéodory, it suffices to show that H_I is Carathéodory.

First, it is easy to see that $H_I(\cdot,\mathbf{p})$ constitutes a well defined conditional expectation of a nonnegative random variable, for all $\mathbf{p} \in \mathcal{S}$. Therefore, what remains is to show that $H_I(\omega,\cdot)$ is continuous on $\mathcal{S}$, everywhere with respect to $\omega \in \Omega$. For this, we will rely on the sequential definition of continuity and the explicit representation of H_I as an integral with respect to the Lebesgue measure, which exploits the form of the projective system of finite dimensional distributions of $|f(\mathbf{p},t)|^2$ and $|g(\mathbf{p},t)|^2$. In particular, because of the trick (3.11), it is easy to show that H_I can be equivalently expressed as the Lebesgue integral

$$
H_I(\omega,\mathbf{p}) = \int_{\mathbb{R}^2} r(\boldsymbol{x})\,\mathcal{N}(\boldsymbol{x};\boldsymbol{\mu}_2(\omega,\mathbf{p}),\boldsymbol{\Sigma}_2(\omega,\mathbf{p}))\,\mathrm{d}\boldsymbol{x}, \tag{8.117}
$$

where the continuous function $r : \mathbb{R}^2 \to \mathbb{R}_{++}$ is defined as (recall that $\varsigma \equiv \log(10)/10$)

$$
r(\boldsymbol{x}) \equiv r(x_1,x_2) \triangleq \frac{P_c P_0 10^{\rho/10} [\exp(x_1+x_2)]^{\varsigma}}{P_0\sigma_D^2 [\exp(x_1)]^{\varsigma} + P_c\sigma^2 [\exp(x_2)]^{\varsigma} + 10^{-\rho/10}\sigma^2\sigma_D^2}, \tag{8.118}
$$

for all $\boldsymbol{x} \equiv (x_1, x_2) \in \mathbb{R}^2$, and $\mathcal{N} : \mathbb{R}^2 \times \mathcal{S} \times \Omega \rightarrow \mathbb{R}_{++}$, corresponds to the *jointly Gaussian* conditional density of $F(\mathbf{p}, t)$ and $G(\mathbf{p}, t)$, relative to $\mathscr{C}(\mathcal{T}_{t-1})$, with mean $\boldsymbol{\mu}_2 : (\omega, \mathbf{p}) \rightarrow \mathbb{R}^{2\times 1}$ and covariance $\boldsymbol{\Sigma}_2 : (\omega, \mathbf{p}) \rightarrow \mathbb{S}^{2\times 2}_{++}$ explicitly depending on ω and $\mathbf{p}$ as

$$\boldsymbol{\mu}_2(\omega, \mathbf{p}) \equiv \boldsymbol{\mu}_2(\mathscr{C}(\mathcal{T}_{t-1})(\omega); \mathbf{p}) \quad \text{and} \tag{8.119}$$

$$\boldsymbol{\Sigma}_2(\omega, \mathbf{p}) \equiv \boldsymbol{\Sigma}_2(\mathscr{C}(\mathcal{T}_{t-1})(\omega); \mathbf{p}), \quad \forall (\omega, \mathbf{p}) \in \Omega \times \mathcal{S}. \tag{8.120}$$

Via a simple change of variables, we may reexpress $H_I(\omega, \mathbf{p})$ as

$$H_I(\omega, \mathbf{p}) \equiv \int_{\mathbb{R}^2} r(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \mathbf{p})) \mathcal{N}(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \mathbf{p})) \, \mathrm{d}\boldsymbol{x}. \tag{8.121}$$

It is straightforward to verify that both $\boldsymbol{\mu}_2(\omega, \cdot)$ and $\boldsymbol{\Sigma}_2(\omega, \cdot)$ are continuous functions in $\mathbf{p} \in \mathcal{S}$, for all $\omega \in \Omega$. This is due to the fact that all functions involving $\mathbf{p}$ in the wireless channel model introduced in Section 3 are trivially continuous in this variable. Equivalently, we may assert that the whole integrand $r(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \cdot)) \mathcal{N}(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \cdot))$ is a continuous function, for all pairs $(\omega, \boldsymbol{x}) \in \Omega \times \mathbb{R}^2$. Next, fix $\omega \in \Omega$, and for *arbitrary* $\mathbf{p} \in \mathcal{S}$, consider *any* sequence $\{\mathbf{p}_k \in \mathcal{S}\}_{k\in\mathbb{N}}$, such that $\mathbf{p}_k \underset{k\to\infty}{\longrightarrow} \mathbf{p}$. Then, $H_I(\omega, \cdot)$ is continuous if and only if $H_I(\omega, \mathbf{p}_k) \underset{k\to\infty}{\longrightarrow} H_I(\omega, \mathbf{p})$. We will show this via a simple application of the Dominated Convergence Theorem. Emphasizing the dependence on $\mathbf{p}$ as a superscript for the sake of clarity, we can write

$$\begin{aligned}
r\left(\boldsymbol{x} + \boldsymbol{\mu}_2^{\mathbf{p}}\right) \mathcal{N}\left(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2^{\mathbf{p}}\right) &\equiv r\left(\boldsymbol{x} + \boldsymbol{\mu}_2^{\mathbf{p}}\right) \frac{\exp\left(-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} \left[\boldsymbol{\Sigma}_2^{\mathbf{p}}\right]^{-1} \boldsymbol{x}\right)}{2\pi\sqrt{\det\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}} \\
&\leq r\left(\boldsymbol{x} + \boldsymbol{\mu}_2^{\mathbf{p}}\right) \frac{\exp\left(-\frac{1}{2}\lambda_{min}\left(\left[\boldsymbol{\Sigma}_2^{\mathbf{p}}\right]^{-1}\right) \|\boldsymbol{x}\|_2^2\right)}{2\pi\sqrt{\det\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}} \\
&\equiv r\left(\boldsymbol{x} + \boldsymbol{\mu}_2^{\mathbf{p}}\right) \frac{\exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2\lambda_{max}\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}\right)}{2\pi\sqrt{\det\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}} \\
&\leq \frac{P_0 10^{\rho/10}}{\sigma^2} \left[\exp\left(x_1 + \boldsymbol{\mu}_2^{\mathbf{p}}(1)\right)\right]^{\varsigma} \frac{\exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2\lambda_{max}\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}\right)}{2\pi\sqrt{\det\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}} \\
&\leq \frac{P_0 10^{\rho/10}}{\sigma^2} \left[\exp\left(x_1 + \sup_{\mathbf{p}\in\mathcal{S}} \boldsymbol{\mu}_2^{\mathbf{p}}(1)\right)\right]^{\varsigma} \frac{\exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2\sup_{\mathbf{p}\in\mathcal{S}}\lambda_{max}\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}\right)}{2\pi\sqrt{\inf_{\mathbf{p}\in\mathcal{S}}\det\left(\boldsymbol{\Sigma}_2^{\mathbf{p}}\right)}} \\
&\triangleq \frac{P_0 10^{\rho/10}}{\sigma^2} \left[\exp(x_1 + p_1)\right]^{\varsigma} \frac{\exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2p_2}\right)}{2\pi\sqrt{p_3}}
\end{aligned}$$

$$\triangleq \psi\left(\omega, \boldsymbol{x}\right), \tag{8.122}$$

where, due to the continuity of $\boldsymbol{\mu}_2(\omega, \cdot)$ and $\boldsymbol{\Sigma}_2(\omega, \cdot)$, the continuity of the maximum eigenvalue and determinant operators, the fact that $\mathcal{S}$ is compact, and the power of the Extreme Value Theorem, all extrema involved are finite and, of course, independent of $\mathbf{p}$. It is now easy to verify that the RHS of (8.122) is integrable. Indeed, by Fubini's Theorem (Theorem 2.6.4 in [45])

$$\begin{aligned}
\int_{\mathbb{R}^2} \psi\left(\omega, \boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} &= \frac{P_0 10^{\rho/10}}{\sigma^2} \frac{\exp(\varsigma p_1)}{\sqrt{p_3}} \int_{\mathbb{R}^2} \exp(\varsigma x_1) \frac{1}{2\pi} \exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2p_2}\right) \mathrm{d}\boldsymbol{x} \\
&\equiv \frac{P_0 10^{\rho/10}}{\sigma^2} \frac{\exp(\varsigma p_1) p_2}{\sqrt{p_3}} \int_{\mathbb{R}^2} \exp(\varsigma x_1) \frac{1}{2\pi p_2} \exp\left(-\frac{\|\boldsymbol{x}\|_2^2}{2p_2}\right) \mathrm{d}\boldsymbol{x} \\
&= \frac{P_0 10^{\rho/10}}{\sigma^2} \frac{\exp(\varsigma p_1) p_2}{\sqrt{p_3}} \int_{\mathbb{R}} \exp(\varsigma x_1) \frac{1}{\sqrt{2\pi p_2}} \exp\left(-\frac{x_1^2}{2p_2}\right) \mathrm{d}x_1 \\
&= \frac{P_0 10^{\rho/10}}{\sigma^2} \frac{\exp(\varsigma p_1(\omega)) p_2(\omega)}{\sqrt{p_3(\omega)}} \exp\left(\frac{p_2(\omega)}{2}\varsigma^2\right) < +\infty, \quad \omega \in \Omega.
\end{aligned} \tag{8.123}$$

That is,

$$\psi\left(\omega, \cdot\right) \in \mathcal{L}_1\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right), \mathcal{L}; \mathbb{R}\right), \quad \omega \in \Omega, \tag{8.124}$$

where $\mathcal{L}$ denotes the Lebesgue measure. We can now call Dominated Convergence; since, for each $\boldsymbol{x} \in \mathbb{R}^2$ (and each $\omega \in \Omega$),

$$r\left(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \mathbf{p}_k)\right) \mathcal{N}\left(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \mathbf{p}_k)\right) \underset{k\to\infty}{\longrightarrow} r\left(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \mathbf{p})\right) \mathcal{N}\left(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \mathbf{p})\right) \tag{8.125}$$

and all members of this sequence are dominated by the integrable function $\psi(\omega, \cdot)$, it is true that

$$\begin{aligned}
H_I(\omega, \mathbf{p}_k) &\equiv \int_{\mathbb{R}^2} r\left(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \mathbf{p}_k)\right) \mathcal{N}\left(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \mathbf{p}_k)\right) \mathrm{d}\boldsymbol{x} \\
&\underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^2} r\left(\boldsymbol{x} + \boldsymbol{\mu}_2(\omega, \mathbf{p})\right) \mathcal{N}\left(\boldsymbol{x}; \mathbf{0}, \boldsymbol{\Sigma}_2(\omega, \mathbf{p})\right) \mathrm{d}\boldsymbol{x} \equiv H_I(\omega, \mathbf{p}).
\end{aligned} \tag{8.126}$$

But $\{\mathbf{p}_k\}_{k\in\mathbb{N}}$ and $\mathbf{p}$ are arbitrary, showing that $H_I(\omega, \cdot)$ is continuous, for each fixed $\omega \in \Omega$. Hence, H_I is Carathéodory on $\Omega \times \mathcal{S}$. ★

The proof to the second part of Theorem 3 follows easily by direct application of the Fundamental Lemma (Lemma 4; also see Table 1). ■

8.3.2 Proof of Lemma 2

In the notation of the statement of the lemma, the joint conditional distribution of $[F(\mathbf{p}, t)\; G(\mathbf{p}, t)]^{\boldsymbol{T}}$ relative to the σ-algebra $\mathscr{C}(\mathcal{T}_{t-1})$ can be readily shown to be Gaussian with mean $\boldsymbol{\mu}_{t|t-1}^{F,G}(\mathbf{p})$ and covariance $\boldsymbol{\Sigma}_{t|t-1}^{F,G}(\mathbf{p})$, for all $(\mathbf{p}, t) \in \mathcal{S} \times \mathbb{N}_{N_T}^2$. This is due to the fact that, in Section 3, we have implicitly assumed that the channel fields $F(\mathbf{p}, t)$ and $G(\mathbf{p}, t)$ are jointly Gaussian. It is then a typical exercise (possibly somewhat tedious though) to show that the functions $\boldsymbol{\mu}_{t|t-1}^{F,G}$ and $\boldsymbol{\Sigma}_{t|t-1}^{F,G}$ are of the form asserted in the statement of the lemma. Regarding the proof for (4.34), observe that we can write

$$\mathbb{E}\left\{ |f(\mathbf{p}, t)|^m |g(\mathbf{p}, t)|^n \middle| \mathscr{C}(\mathcal{T}_{t-1}) \right\}$$

$$
\begin{aligned}
&\equiv 10^{(m+n)\rho/20}\mathbb{E}\left\{\exp\left(\frac{\log(10)}{20}\left(mF(\mathbf{p},t)+nG(\mathbf{p},t)\right)\right)\middle|\mathscr{C}(\mathcal{T}_{t-1})\right\} \\
&\equiv 10^{(m+n)\rho/20}\mathbb{E}\left\{\exp\left(\frac{\log(10)}{20}[m\,n]\left[F(\mathbf{p},t)\;G(\mathbf{p},t)\right]^{\boldsymbol{T}}\right)\middle|\mathscr{C}(\mathcal{T}_{t-1})\right\},
\end{aligned} \tag{8.127}
$$

with the conditional expectation on the RHS being nothing else than the conditional moment generating function of the conditionally jointly Gaussian random vector $[F(\mathbf{p},t)\;G(\mathbf{p},t)]^{\boldsymbol{T}}$ at each $\mathbf{p}$ and t, evaluated at the point $(\log(10)/20)[m\,n]^{\boldsymbol{T}}$, for any choice of $(m,n)\in\mathbb{Z}\times\mathbb{Z}$. Recalling the special form of the moment generating function for Gaussian random vectors, the result readily follows. ■

8.3.3 Proof of Theorem 4

It will suffice to show that both objectives of (4.35) and (4.36) are Carathéodory in $\Omega\times\mathcal{S}$. But this statement may be easily shown by analytically expressing both (4.35) and (4.36) using Lemma 2. Now, since both objectives of (4.35) and (4.36) are Carathéodory, we may invoke Theorem 10 (Appendix B), in an inductive fashion, for each $t\in\mathbb{N}^2_{N_T}$, guaranteeing the existence of at least one $\mathscr{C}(\mathcal{T}_{t-1})$-measurable decision for either (4.35), or (4.36), say $\widetilde{\mathbf{p}}^*(t)$, which solves the optimization problem considered, for all $\omega\in\Omega$. Proceeding inductively gives the result. ■

8.3.4 Proof of Theorem 6

By assumption, $V(\mathbf{p},t)$ is $\mathbf{L.MD.G}\diamondsuit(\mathscr{H}_t,\mu)$, implying, for every $t\in\mathbb{N}^+_{N_T}$, the existence of an event $\Omega_t\subseteq\Omega$, satisfying $\mathcal{P}(\Omega_t)\equiv 1$, such that, for every $\mathbf{p}\in\mathcal{S}^R$,

$$
\mu\mathbb{E}\left\{V(\mathbf{p},t-1)|\,\mathscr{H}_{t-1}\right\}(\omega)\equiv\mathbb{E}\left\{V(\mathbf{p},t)|\,\mathscr{H}_{t-1}\right\}(\omega),\quad\forall\omega\in\Omega_t. \tag{8.128}
$$

Fix $t\in\mathbb{N}^2_{N_T}$. Consider any admissible policy $\mathbf{p}^o(t)$ at t, implemented at t and decided at $t-1\in\mathbb{N}^+_{N_T-1}$. By our assumptions, $V(\cdot,\cdot,t)$ is $\boldsymbol{SP}\diamondsuit\mathfrak{C}_{\mathscr{H}_t}$. Additionally, because $\mathbf{p}^o(t)$ is admissible, it will be measurable relative to the limit σ-algebra $\mathscr{P}^{\uparrow}_t$ and, hence, measurable relative to $\mathscr{H}_t$. Thus, there exists an event $\Omega^{\mathbf{p}^o}_t\subseteq\Omega$, with $\mathcal{P}\left(\Omega^{\mathbf{p}^o}_t\right)\equiv 1$, such that, for every $\omega\in\Omega^{\mathbf{p}^o}_t$,

$$
\begin{aligned}
\mathbb{E}\left\{V(\mathbf{p}^o(t),t)|\,\mathscr{H}_t\right\}(\omega)&\equiv\mathbb{E}\left\{V(\mathbf{p},t)|\,\mathscr{H}_t\right\}(\omega)|_{\mathbf{p}=\mathbf{p}^o(\omega,t)} \\
&\equiv h_t(\omega,\mathbf{p}^o(\omega,t)),
\end{aligned} \tag{8.129}
$$

where the extended real-valued random function $h_t:\Omega\times\mathcal{S}^R\to\overline{\mathbb{R}}$ is jointly $\mathscr{H}_t\otimes\mathscr{B}\left(\mathcal{S}^R\right)$-measurable, with $h_t(\omega,\mathbf{p})\equiv\mathbb{E}\left\{V(\mathbf{p},t)|\,\mathscr{H}_t\right\}(\omega)$, everywhere in $(\omega,\mathbf{p})\in\Omega\times\mathcal{S}^R$.

Also by our assumptions, $V(\cdot,\cdot,t)$ is $\boldsymbol{SP}\diamondsuit\mathfrak{C}_{\mathscr{H}_{t-1}}$, as well. Similarly to the arguments made above, if $\mathbf{p}^o(t)$ is assumed to be measurable relative to the limit σ-algebra $\mathscr{P}^{\uparrow}_{t-1}$, or, in other words, admissible at $t-1$, then it will also be measurable relative to $\mathscr{H}_{t-1}$. Therefore, there exists an event $\Omega^{\mathbf{p}^o}_{t^-}\subseteq\Omega$, with $\mathcal{P}\left(\Omega^{\mathbf{p}^o}_{t^-}\right)\equiv 1$, such that, for every $\omega\in\Omega^{\mathbf{p}^o}_{t^-}$,

$$
\begin{aligned}
\mathbb{E}\left\{V(\mathbf{p}^o(t),t)|\,\mathscr{H}_{t-1}\right\}(\omega)&\equiv\mathbb{E}\left\{V(\mathbf{p},t)|\,\mathscr{H}_{t-1}\right\}(\omega)|_{\mathbf{p}=\mathbf{p}^o(\omega,t)} \\
&\equiv h_{t^-}(\omega,\mathbf{p}^o(\omega,t)),
\end{aligned} \tag{8.130}
$$

where the random function $h_{t^-} : \Omega \times \mathcal{S}^R \rightarrow \overline{\mathbb{R}}$ is jointly $\mathscr{H}_{t-1} \otimes \mathscr{B}\left(\mathcal{S}^R\right)$-measurable, with $h_{t^-}(\omega, \mathbf{p}) \equiv \mathbb{E}\{V(\mathbf{p}, t) | \mathscr{H}_{t-1}\}(\omega)$, everywhere in $(\omega, \mathbf{p}) \in \Omega \times \mathcal{S}^R$. Note that, by construction, $\mathbf{p}^o(t)$ will also be admissible at time t and, therefore, measurable relative to and $\mathscr{H}_t$, as well.

Now, we combine the arguments made above. Keep $t \in \mathbb{N}^2_{N_T}$ fixed. At time slot $t-2 \in \mathbb{N}_{N_T-2}$, let $\mathbf{p}^o(t-1) \equiv \mathbf{p}^o(\omega, t-1)$ be a $\mathscr{C}(\mathcal{T}_{t-2})$-measurable admissible policy (recall that **C1-C6** are satisfied by assumption; also recall that, if $t \equiv 2$, $\mathscr{C}(\mathcal{T}_{t-2}) \equiv \mathscr{C}(\mathcal{T}_0)$ is the trivial σ-algebra). At the *next* time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, let us choose $\mathbf{p}^o(t) \equiv \mathbf{p}^o(\omega, t-1)$; in this case, $\mathbf{p}^o(t)$ will also be $\mathscr{C}(\mathcal{T}_{t-2})$-measurable and result in the *same final position for the relays at time slot* $t \in \mathbb{N}^2_{N_T}$. As a result, the relays just stay still. Under these circumstances, at time slot $t-1 \in \mathbb{N}^+_{N_T-1}$, the expected network QoS will be $\mathbb{E}\{V(\mathbf{p}^o(t-1), t-1)\}$, whereas, at the next time slot $t \in \mathbb{N}^2_{N_T}$, it will be $\mathbb{E}\{V(\mathbf{p}^o(t-1), t)\}$. Exploiting (8.128), we may write

$$\mu h_{t-1}(\omega, \mathbf{p}) \equiv h_{t^-}(\omega, \mathbf{p}), \quad \forall (\omega, \mathbf{p}) \in \Omega_t \bigcap \Omega^{\mathbf{p}^o}_{t-1} \bigcap \Omega^{\mathbf{p}^o}_{t^-} \times \mathcal{S}^R, \tag{8.131}$$

where, obviously, $\mathcal{P}\left(\Omega_t \bigcap \Omega^{\mathbf{p}^o}_{t-1} \bigcap \Omega^{\mathbf{p}^o}_{t^-}\right) \equiv 1$. Consequently, it will be true that

$$\mu h_{t-1}(\omega, \mathbf{p}^o(\omega, t-1)) \equiv h_{t^-}(\omega, \mathbf{p}^o(\omega, t-1)), \quad \forall \omega \in \Omega_t \bigcap \Omega^{\mathbf{p}^o}_{t-1} \bigcap \Omega^{\mathbf{p}^o}_{t^-}. \tag{8.132}$$

From (8.129) and (8.130), it is also true that

$$\mu \mathbb{E}\{V(\mathbf{p}^o(t-1), t-1) | \mathscr{H}_{t-1}\}(\omega) \equiv \mathbb{E}\{V(\mathbf{p}^o(t-1), t) | \mathscr{H}_{t-1}\}(\omega), \tag{8.133}$$

almost everywhere with respect to $\mathcal{P}$. This, of course, implies that

$$\mu \mathbb{E}\{V(\mathbf{p}^o(t-1), t-1)\} \equiv \mathbb{E}\{V(\mathbf{p}^o(t-1), t)\}, \tag{8.134}$$

and for all $t \in \mathbb{N}^2_{N_T}$, since t was arbitrary.

Since (8.134) holds for all admissible policies decided at time slot $t-2 \in \mathbb{N}_{N_T-2}$, it will also hold for the respective optimal policy, that is,

$$\mu \mathbb{E}\left\{V\left(\mathbf{p}^*(t-1), t-1\right)\right\} \equiv \mathbb{E}\left\{V\left(\mathbf{p}^*(t-1), t\right)\right\}, \quad \forall t \in \mathbb{N}^2_{N_T}. \tag{8.135}$$

Next, as discussed above, the choice $\mathbf{p}^o(t) \equiv \mathbf{p}^*(\omega, t-1)$ constitutes an admissible policy decided at time slot $t-1 \in \mathbb{N}^+_{N_T-1}$; it suffices to see that $\mathbf{p}^*(\omega, t-1) \in \mathcal{C}\left(\mathbf{p}^*(\omega, t-1)\right)$, by definition of our initial 2-stage problem, because "staying still" is always a feasible decision for the relays. Consequently, because the optimal policy $\mathbf{p}^*(t)$ results in the highest network QoS, *among all admissible policies*, it will be true that

$$\mu \mathbb{E}\left\{V\left(\mathbf{p}^*(t-1), t-1\right)\right\} \leq \mathbb{E}\left\{V\left(\mathbf{p}^*(t), t\right)\right\}, \quad \forall t \in \mathbb{N}^2_{N_T}, \tag{8.136}$$

completing the proof of Theorem 6. ■

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