

Asymptotically Optimal Discrete Time Nonlinear Filters From Stochastically Convergent State Process Approximations

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Abstract

We consider the problem of approximating optimal in the Minimum Mean Squared Error (MMSE) sense nonlinear filters in a *discrete time* setting, exploiting properties of stochastically convergent state process approximations. More specifically, we consider a class of nonlinear, partially observable stochastic systems, comprised by a (possibly nonstationary) hidden stochastic process (the state), observed through another conditionally Gaussian stochastic process (the observations). Under general assumptions, we show that, given an approximating process which, for each time step, is stochastically convergent to the state process, an approximate filtering operator can be defined, which converges to the true optimal nonlinear filter of the state in a strong and well defined sense. In particular, the convergence is compact in time and uniform in a *completely characterized* measurable set of probability measure almost unity, also providing a purely quantitative justification of Egoroff's Theorem for the problem at hand. The results presented in this paper can form a common basis for the analysis and characterization of a number of heuristic approaches for approximating optimal nonlinear filters, such as approximate grid based techniques, known to perform well in a variety of applications.

Keywords. Approximate Nonlinear Filtering, Hidden Models, Partially Observable Systems, Stochastic Processes, \mathcal{C} -Weak Convergence, Change of Probability Measures.

1 Introduction

Nonlinear stochastic filtering refers to problems in which a stochastic process, usually called the state, is partially observed as a result of measuring another stochastic process, usually called the observations or measurements, and the objective is to estimate the state or some functional of it, based only on past and present observations. The nonlinearity is due to the general, possibly non Gaussian nature of the state and observations processes, as well as the fact that, in general, the state may be partially observed as a nonlinear functional of the observations. Usually, nonlinear state estimators are designed so as to optimize some performance criterion. Most commonly, this corresponds to the Minimum Mean Squared Error (MMSE), which is also adopted in this work.

A desirable feature of a nonlinear filter is *recursiveness* in time, as it greatly reduces computational complexity and allows for real time estimation as new measurements become available.

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However, not all nonlinear filters possess this important property [2, 3]. Recursive nonlinear filters exist for some very special cases, such as those in which the transition model of the state process is linear (Gauss-Markov), or when the state is a Markov chain (discrete state space) [4–7]. In the absence of recursive filter representations, practical filtering schemes have been developed, which typically approximate the desired quantities of interest, either heuristically (e.g., Gaussian approximations [8, 9]) or in some more powerful, rigorous sense (e.g., Markov chain approximations [10, 11]).

In this paper, we follow the latter research direction. Specifically, we consider a partially observable system in *discrete time*, comprised by a hidden, almost surely compactly bounded state process, observed through another, conditionally Gaussian measurement process. The mean and covariance matrix of the measurements both constitute nonlinear, time varying and state dependent functions, assumed to be known a priori. Employing a change of measure argument and using the original measurements, an approximate filtering operator can be defined, by replacing the “true” state process by an appropriate approximation. Our contribution is summarized in showing that if the approximation converges to the state either in probability or in the \mathcal{C} -weak sense (Section II.C), the resulting filtering operator converges to the true optimal nonlinear filter in a relatively strong and well defined sense; the convergence is compact in time and uniform in a measurable set of probability measure almost unity (Theorem 3). The aforementioned set is completely characterized in terms of a subset of the parameters of the filtering problem of interest. Consequently, our results provide a purely quantitative justification of Egoroff’s theorem [12] for the problem at hand, which concerns the equivalence of almost sure convergence and almost uniform convergence of measurable functions.

To better motivate the reader, let us describe two problems that fit the scenario described above and can benefit from the contributions of this paper, namely, those of *sequential channel state estimation* and *(sequential) spatiotemporal channel prediction* [13] (see also [14]). The above problems arise naturally in novel signal processing applications in the emerging area of distributed, autonomous, physical layer aware mobile networks [15–17]. Such networks usually consist of cooperating mobile sensors, each of them being capable of observing its communication channel (under a flat fading assumption), relative to a reference point in the space. In most practical scenarios, the dominant quantities characterizing the wireless links, such as the path loss exponent and the shadowing power, behave as stochastic processes themselves. For instance, such behavior may be due to physical changes in the environment and also the inherent randomness of the communication medium itself. Then, the path loss exponent and the shadowing power can be collectively considered as the hidden state (suggestively called the channel state) of a partially observable system, where the channel gains measured at each sensor can be considered as the corresponding observations. In general, such observations are nonlinear functionals of the state. Assuming additionally that the channel state is a Markov process, the main results presented herein can essentially provide strong asymptotic guarantees for approximate sequential nonlinear channel state estimation and spatiotemporal channel prediction, enabling physical layer aware motion planning and stochastic control. For more details, the reader is referred to [13].

The idea of replacing the process of interest with some appropriate approximation is borrowed from [11]. However, [11] deals almost exclusively with continuous time stochastic systems and the results presented in there do not automatically extend to the discrete time system setting we are dealing with here. In fact, the continuous time counterparts of the discrete time stochastic processes considered here are considerably more general than the ones treated in [11]. More specifically, although some relatively general results are indeed provided for continuous time hidden processes,

[11] is primarily focused on the standard hidden diffusion case, which constitutes a Markov process (and aiming to the development of recursive approximate filters), whereas, in our setting, the hidden process is initially assumed to be arbitrary (as long as it is confined to a compact set). Also, different from our formulation (see above), in [11], the covariance matrix of the observation process *does not* depend on the hidden state; the state affects only the mean of the observations. Further, the modes of stochastic convergence considered here are different compared to [11] (in fact, they are stronger), both regarding convergence of approximations and convergence of approximate filters.

The results presented in this paper provide a framework for analyzing a number of heuristic techniques for numerically approximating optimal nonlinear filters in discrete time, such as approximate grid based recursive approaches, known to perform well in a wide variety of applications [13, 18, 19]. Additionally, our results do not refer exclusively to recursive nonlinear filters. The sufficient conditions which we provide for the convergence of approximate filtering operators are independent of the way a filter is realized (see Section III). This is useful because, as highlighted in [20], no one prevents one from designing an efficient (approximate) nonlinear filter which is part recursive and part nonrecursive, or even possibly trying to combine the best of both worlds, and there are practical filters designed in this fashion [20].

The paper is organized as follows: In Section II, we introduce the system model, along with some mild technical assumptions on its structure and also present/develop some preliminary technical results and definitions, which are important for stating and proving our results. In Section III, we formulate our problem in detail and present our main results (Theorem (3)), along with a simple instructive example. Section IV is exclusively devoted to proving the results stated in Section III. Finally, Section V concludes the paper.

Notation: In the following, the state vector will be represented as X_t , its approximations as $X_t^{L_S}$, and all other matrices and vectors, either random or not, will be denoted by boldface letters (to be clear by the context). Real valued random variables and abstract random elements will be denoted by uppercase letters. Calligraphic letters and formal script letters will denote sets and σ -algebras, respectively. The operators $(\cdot)^T$, $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ will denote transposition, minimum and maximum eigenvalue, respectively. For any random element (same for variable, vector) Y , $\sigma\{Y\}$ will denote the σ -algebra generated by Y . The ℓ_p norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is $\|\mathbf{x}\|_p \triangleq (\sum_{i=1}^n |x_i|^p)^{1/p}$, for all naturals $p \geq 1$. The spectral and Frobenius norms of any matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$ are $\|\mathbf{X}\|_2 \triangleq \max_{\|\mathbf{x}\|_2=1} \|\mathbf{X}\mathbf{x}\|_2$ and $\|\mathbf{X}\|_F \triangleq \sqrt{\sum_{i,j=1}^{n,n} |\mathbf{X}_{ij}|^2}$, respectively. Positive definiteness and semidefiniteness of \mathbf{X} will be denoted by $\mathbf{X} \succ \mathbf{0}$ and $\mathbf{X} \succeq \mathbf{0}$, respectively. For any Euclidean space $\mathbb{R}^{N \times 1}$, $\mathbf{I}_{N \times N}$ will denote the respective identity operator. Additionally, throughout the paper, we employ the identifications $\mathbb{R}_+ \equiv [0, \infty)$, $\mathbb{R}_{++} \equiv (0, \infty)$, $\mathbb{N}^+ \equiv \{1, 2, \dots\}$, $\mathbb{N}_n^+ \equiv \{1, 2, \dots, n\}$ and $\mathbb{N}_n \equiv \{0\} \cup \mathbb{N}_n^+$, for any positive natural n .

2 Partially Observable System Model & Technical Preliminaries

In this section, we give a detailed description of the partially observable (or hidden) system model of interest and present our related technical assumptions on its components. Additionally, we present some essential background on the measure theoretic concept of change of probability measures and state some definitions and known results regarding specific modes of stochastic convergence, which will be employed in our subsequent theoretical developments.

2.1 Hidden Model: Definitions & Technical Assumptions

First, let us set the basic probabilistic framework, as well as precisely define the hidden system model considered throughout the paper:

- All stochastic processes considered below are fundamentally generated on a common complete probability space (the base space), defined by a triplet $(\Omega, \mathcal{F}, \mathcal{P})$, at each time instant taking values in a measurable state space, consisting of some Euclidean subspace and the associated Borel σ -algebra on that subspace. For example, for each $t \in \mathbb{N}$, the state process $X_t \equiv X_t(\omega)$, where $\omega \in \Omega$, takes its values in the measurable state space $(\mathbb{R}^{M \times 1}, \mathcal{B}(\mathbb{R}^{M \times 1}))$, where $\mathcal{B}(\mathbb{R}^{M \times 1})$ constitutes the Borel σ -algebra of measurable subsets of $\mathbb{R}^{M \times 1}$.
- In this work, the evolution mechanism of state process X_t is assumed to be arbitrary. However, in order to avoid unnecessary technical complications, we assume that, for each $t \in \mathbb{N}$, the induced probability measure of X_t is absolutely continuous with respect to the Lebesgue measure on its respective state space. Then, by the Radon-Nikodym Theorem, it admits a density, unique up to sets of zero Lebesgue measure. Also, we will generically assume that for all $t \in \mathbb{N}$, $X_t \in \mathcal{Z}$ almost surely, where \mathcal{Z} constitutes a compact strict subset of $\mathbb{R}^{M \times 1}$. In what follows, however, in order to lighten the presentation, we will assume that $M \equiv 1$. Nevertheless, all stated results hold with the same validity if $M > 1$ (See also Assumption 2 below).
- The state X_t is partially observed through the observation process

$$\mathbf{y}_t \triangleq \boldsymbol{\mu}_t(X_t) + \boldsymbol{\sigma}_t(X_t) + \boldsymbol{\xi}_t \in \mathbb{R}^{N \times 1}, \quad \forall t \in \mathbb{N}, \quad (1)$$

where, *conditioned on* X_t and for each $t \in \mathbb{N}$, the sequence $\left\{ \boldsymbol{\mu}_t : \mathcal{Z} \mapsto \mathbb{R}^{N \times 1} \right\}_{t \in \mathbb{N}}$ is known apriori, the process $\boldsymbol{\sigma}_t(X_t) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_t(X_t) \succ \mathbf{0})$ constitutes Gaussian noise, with the sequence $\left\{ \boldsymbol{\Sigma}_t : \mathcal{Z} \mapsto \mathcal{D}_{\boldsymbol{\Sigma}} \right\}_{t \in \mathbb{N}}$, where $\mathcal{D}_{\boldsymbol{\Sigma}}$ is a bounded subset of $\mathbb{R}^{N \times N}$, also known apriori, and $\boldsymbol{\xi}_t \stackrel{i.i.d.}{\sim} \mathcal{N}(\mathbf{0}, \sigma_{\xi}^2 \mathbf{I}_{N \times N})$.

As a pair, the state X_t and the observations process described by (1) define a very wide family of partially observable systems. In particular, any Hidden Markov Model (HMM) of *any* order, in which the respective Markov state process is almost surely confined in a compact subset of its respective Euclidean state space, is indeed a member of this family. More specifically, let us rewrite (1) in the canonical form

$$\mathbf{y}_t \equiv \boldsymbol{\mu}_t(X_t) + \sqrt{\mathbf{C}_t(X_t)} \mathbf{u}_t \in \mathbb{R}^{N \times 1}, \quad \forall t \in \mathbb{N}, \quad (2)$$

where $\mathbf{u}_t \equiv \mathbf{u}_t(\omega)$ constitutes a standard Gaussian white noise process and, for all $x \in \mathcal{Z}$, $\mathbf{C}_t(x) \triangleq \boldsymbol{\Sigma}_t(x) + \sigma_{\xi}^2 \mathbf{I}_{N \times N} \in \mathcal{D}_{\mathbf{C}}$, with $\mathcal{D}_{\mathbf{C}}$ bounded. Then, for a possibly nonstationary HMM of order m , assuming the existence of an explicit functional model for describing the temporal evolution of the state (being a Markov process of order m), we get the system of *standardized* stochastic difference equations

$$\begin{aligned} X_t &\equiv f_t \left(\{X_{t-i}\}_{i \in \mathbb{N}_m^+}, \mathbf{W}_t \right) \in \mathcal{Z}, \\ \mathbf{y}_t &\equiv \boldsymbol{\mu}_t(X_t) + \sqrt{\mathbf{C}_t(X_t)} \mathbf{u}_t \end{aligned}, \quad \forall t \in \mathbb{N}, \quad (3)$$

where, for each t , $f_t : \mathcal{Z}^m \times \mathcal{W} \xrightarrow{a.s.} \mathcal{Z}$ (with $\mathcal{Z}^m \triangleq \times_{m \text{ times}} \mathcal{Z}$) constitutes a measurable nonlinear state transition mapping and $\mathbf{W}_t \equiv \mathbf{W}_t(\omega) \in \mathcal{W} \subseteq \mathbb{R}^{M_{\mathcal{W}} \times 1}$, denotes a (discrete time) white noise process with state space \mathcal{W} . For a first order stationary HMM, the above system of equations reduces to

$$\begin{aligned} X_t &\equiv f(X_{t-1}, \mathbf{W}_t) \in \mathcal{Z} \\ \mathbf{y}_t &\equiv \boldsymbol{\mu}_t(X_t) + \sqrt{\mathbf{C}_t(X_t)} \mathbf{u}_t \end{aligned}, \quad \forall t \in \mathbb{N}, \quad (4)$$

which arguably constitutes the most typical partially observable system model encountered in both Signal Processing and Control, with plethora of important applications.

Let us also present some more specific assumptions, regarding the nature (boundedness, continuity and expansiveness) of the aforementioned sequences of functions.

Assumption 1: (Boundedness) For later reference, let

$$\lambda_{inf} \triangleq \inf_{t \in \mathbb{N}} \inf_{x \in \mathcal{Z}} \lambda_{min}(\mathbf{C}_t(x)), \quad (5)$$

$$\lambda_{sup} \triangleq \sup_{t \in \mathbb{N}} \sup_{x \in \mathcal{Z}} \lambda_{max}(\mathbf{C}_t(x)), \quad (6)$$

$$\mu_{sup} \triangleq \sup_{t \in \mathbb{N}} \sup_{x \in \mathcal{Z}} \|\boldsymbol{\mu}_t(x)\|_2, \quad (7)$$

where each quantity of the above is uniformly and finitely bounded for all $t \in \mathbb{N}$ and for all $x \in \mathcal{Z}$. If x is substituted by the stochastic process $X_t(\omega)$, then all the above definitions continue to hold in the essential sense. For technical reasons related to the bounding-from-above arguments presented in Section IV, containing the proof of the main result of the paper, it is also assumed that $\lambda_{inf} > 1$, a requirement which can always be satisfied by appropriate normalization of the observations.

Assumption 2: (Continuity & Expansiveness) All members of the functional family $\left\{ \boldsymbol{\mu}_t : \mathcal{Z} \mapsto \mathbb{R}^{N \times 1} \right\}_{t \in \mathbb{N}}$ are uniformly Lipschitz continuous, that is, there exists a bounded constant $K_\mu \in \mathbb{R}_+$, such that, for all $t \in \mathbb{N}$,

$$\|\boldsymbol{\mu}_t(x) - \boldsymbol{\mu}_t(y)\|_2 \leq K_\mu |x - y|, \quad \forall (x, y) \in \mathcal{Z} \times \mathcal{Z}. \quad (8)$$

Additionally, all members of the functional family $\left\{ \boldsymbol{\Sigma}_t : \mathcal{Z} \mapsto \mathcal{D}_{\boldsymbol{\Sigma}} \subset \mathbb{R}^{N \times N} \right\}_{t \in \mathbb{N}}$ are *elementwise* uniformly Lipschitz continuous, that is, there exists some universal and bounded constant $K_{\boldsymbol{\Sigma}} \in \mathbb{R}_+$, such that, for all $t \in \mathbb{N}$ and for all $(i, j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+$,

$$\left| \boldsymbol{\Sigma}_t^{ij}(x) - \boldsymbol{\Sigma}_t^{ij}(y) \right| \leq K_{\boldsymbol{\Sigma}} |x - y|, \quad \forall (x, y) \in \mathcal{Z} \times \mathcal{Z}. \quad (9)$$

If x is substituted by the stochastic process $X_t(\omega)$, then all the above statements are understood in the almost sure sense.

Remark 1. As we have already said, for simplicity, we assume that $\mathcal{Z} \subset \mathbb{R}$, that is, $M \equiv 1$. In any other case (when $M > 1$), we modify the Lipschitz assumptions stated above simply by replacing $|x - y|$ with $\|\mathbf{x} - \mathbf{y}\|_1$, that is, Lipschitz continuity is meant to be with respect to the ℓ_1 norm in the domain of the respective function. If this holds, everything that follows works also in $\mathbb{R}^{M > 1}$, just with some added complexity in the proofs of the results. Also, because $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ for any $\mathbf{x} \in \mathbb{R}^M$, the assumed Lipschitz continuity with respect to ℓ_1 norm can be replaced by Lipschitz continuity with respect to the ℓ_2 norm, since the latter implies the former, and again everything holds. Further, if $M > 1$, convergence in probability and \mathcal{L}_1 convergence of random vectors are both defined by replacing absolute values with the ℓ_1 norms of the vectors under consideration. ■

2.2 Conditional Expectations, Change of Measure & Filters

Before proceeding with the general formulation of our estimation problem and for later reference, let us define the complete natural filtrations of the processes X_t and \mathbf{y}_t as

$$\{\mathcal{X}_t\}_{t \in \mathbb{N}} \triangleq \left\{ \sigma \left\{ \{X_i\}_{i \in \mathbb{N}_t} \right\} \right\}_{t \in \mathbb{N}} \quad \text{and} \quad (10)$$

$$\{\mathcal{Y}_t\}_{t \in \mathbb{N}} \triangleq \left\{ \sigma \left\{ \{\mathbf{y}_i\}_{i \in \mathbb{N}_t} \right\} \right\}_{t \in \mathbb{N}}, \quad (11)$$

respectively, and also the complete filtration generated by both X_t and \mathbf{y}_t as

$$\{\mathcal{H}_t\}_{t \in \mathbb{N}} \triangleq \left\{ \sigma \left\{ \{X_i, \mathbf{y}_i\}_{i \in \mathbb{N}_t} \right\} \right\}_{t \in \mathbb{N}}. \quad (12)$$

In all the above, $\sigma\{Y\}$ denotes the σ -algebra generated by the random variable Y .

In this work, we adopt the MMSE as an optimality criterion. In this case, one would ideally like to discover a solution to the stochastic optimization problem

$$\begin{aligned} \inf_{\hat{X}_t} \quad & \mathbb{E} \left\{ \left\| X_t - \hat{X}_t \right\|_2^2 \right\}, \quad \forall t \in \mathbb{N}, \\ \text{subject to} \quad & \mathbb{E} \left\{ \hat{X}_t \mid \mathcal{Y}_t \right\} \equiv \hat{X}_t \end{aligned} \quad (13)$$

where the constraint is equivalent to confining the search for possible estimators \hat{X}_t to the subset of interest, that is, containing the ones which constitute \mathcal{Y}_t -measurable random variables. Of course, the solution to the program (13) coincides with the conditional expectation [21]

$$\mathbb{E} \{ X_t \mid \mathcal{Y}_t \} \equiv \hat{X}_t, \quad \forall t \in \mathbb{N}, \quad (14)$$

which, in the nonlinear filtering literature, is frequently called a *filter*. There is also an alternative and very useful way of reexpressing the filter process \hat{X}_t , using the concept of *change of probability measures*, which will allow us to stochastically decouple the state and observations of our hidden system and then let us formulate precisely the approximation problem of interest in this paper. Change of measure techniques have been extensively used in discrete time nonlinear filtering, mainly in order to discover recursive representations for various hidden Markov models [3, 6, 7, 22]. In the following, we provide a brief introduction to these type of techniques (suited to our purposes) which is also intuitive, simple and technically accessible, including direct proofs of the required results.

Change of Probability Measure in Discrete Time: Demystification & Useful Results

So far, all stochastic processes we have considered are defined on the base space $(\Omega, \mathcal{F}, \mathcal{P})$. In fact, it is the structure of the probability measure \mathcal{P} that is responsible for the coupling between the stochastic processes X_t and \mathbf{y}_t , being, for each $t \in \mathbb{N}$, measurable functions from (Ω, \mathcal{F}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$, respectively. Intuitively, the measure \mathcal{P} constitutes our “reference measurement tool” for measuring the events contained in the base σ -algebra \mathcal{F} , and any random variable serves as a “medium” or “channel” for observing these events.

As a result, some very natural questions arise from the above discussion. First, one could ask if and under what conditions it is possible to change the probability measure \mathcal{P} , which constitutes our *fixed* way of assigning probabilities to events, to another measure $\tilde{\mathcal{P}}$ on the same measurable space (Ω, \mathcal{F}) , in a way such that there exists some sort of transformation connecting \mathcal{P} and $\tilde{\mathcal{P}}$.

Second, if we can indeed make the transition from \mathcal{P} to $\tilde{\mathcal{P}}$, could we choose the latter probability measure in a way such that the processes X_t and \mathbf{y}_t behave according to a prespecified statistical model? For instance, we could demand that, under \mathcal{P} , X_t and \mathbf{y}_t constitute independent stochastic processes. Third and most important, is it possible to derive an expression for the “original” filter $\hat{X}_t \equiv \mathbb{E}_{\mathcal{P}}\{X_t|\mathcal{Y}_t\}$ under measure \mathcal{P} , using only (conditional) expectations under $\tilde{\mathcal{P}}$ (denoted as $\mathbb{E}_{\tilde{\mathcal{P}}}\{\cdot|\cdot\}$)?

The answers to all three questions stated above are affirmative under very mild assumptions and the key result in order to prove this assertion is the Radon-Nikodym Theorem [23]. However, assuming that the induced joint probability measure of the processes of interest is absolutely continuous with respect to the Lebesgue measure of the appropriate dimension, in the following we provide an answer to these questions, employing only elementary probability theory, avoiding the direct use of the Radon-Nikodym Theorem.

Theorem 1. (Conditional Bayes’ Theorem for Densities) *Consider the (possibly vector) stochastic processes $X_t(\omega) \in \mathbb{R}^{N_t \times 1}$ and $Y_t(\omega) \in \mathbb{R}^{M_t \times 1}$, both defined on the same measurable space (Ω, \mathcal{F}) , for all $t \in \mathbb{N}$. Further, if \mathcal{P} and $\tilde{\mathcal{P}}$ are two probability measures on (Ω, \mathcal{F}) , suppose that:*

- Under both \mathcal{P} and $\tilde{\mathcal{P}}$, the process X_t is integrable.
- Under the base probability measure \mathcal{P} (resp. $\tilde{\mathcal{P}}$), the induced joint probability measure of $(\{X_i\}_{i \in \mathbb{N}_t}, \{Y_i\}_{i \in \mathbb{N}_t})$ is absolutely continuous with respect to the Lebesgue measure of the appropriate dimension, implying the existence of a density f_t (resp. \tilde{f}_t), with

$$f_t : \left(\prod_{i \in \mathbb{N}_t} \mathbb{R}^{N_i \times 1} \right) \times \left(\prod_{i \in \mathbb{N}_t} \mathbb{R}^{M_i \times 1} \right) \mapsto \mathbb{R}_+. \quad (15)$$

- For each set of points, it is true that

$$\tilde{f}_t(\cdots) \equiv 0 \quad \Rightarrow \quad f_t(\cdots) \equiv 0, \quad (16)$$

or, equivalently, the support of f_t is contained in the support of \tilde{f}_t .

Also, for all $t \in \mathbb{N}$, define the Likelihood Ratio (LR) at t as the $\{\mathcal{H}_t\}$ -adapted, nonnegative stochastic process¹

$$\Lambda_t \triangleq \frac{f_t(X_0, X_1, \dots, X_t, Y_0, Y_1, \dots, Y_t)}{\tilde{f}_t(X_0, X_1, \dots, X_t, Y_0, Y_1, \dots, Y_t)}. \quad (17)$$

Then, it is true that

$$\hat{X}_t \equiv \mathbb{E}_{\mathcal{P}}\{X_t|\mathcal{Y}_t\} \equiv \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \Lambda_t|\mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\Lambda_t|\mathcal{Y}_t\}}, \quad (18)$$

almost everywhere with respect to \mathcal{P} .

Proof of Theorem 1. See the Appendix. ■

¹With zero probability of confusion, we use $\{\mathcal{Y}_t\}_{t \in \mathbb{N}}$ and $\{\mathcal{H}_t\}_{t \in \mathbb{N}}$ to denote the complete filtrations generated by Y_t and $\{X_t, Y_t\}$.

Remark 2. The $\{\mathcal{H}_t\}$ -adapted LR process

$$\Lambda_t \equiv \Lambda_t \left(\mathcal{X}_t \triangleq \{X_i\}_{i \in \mathbb{N}_t}, \mathcal{Y}_t \triangleq \{Y_i\}_{i \in \mathbb{N}_t} \right), \quad t \in \mathbb{N}, \quad (19)$$

as defined in (17), actually coincides with the restriction of the Radon-Nikodym derivative of \mathcal{P} with respect to $\tilde{\mathcal{P}}$ to the filtration $\{\mathcal{H}_t\}_{t \in \mathbb{N}}$, that is,

$$\left. \frac{d\mathcal{P}(\omega)}{d\tilde{\mathcal{P}}(\omega)} \right|_{\mathcal{H}_t} \equiv \Lambda_t(\mathcal{X}_t(\omega), \mathcal{Y}_t(\omega)), \quad \forall t \in \mathbb{N}, \quad (20)$$

a statement which, denoting the collections $\{x_i\}_{i \in \mathbb{N}_t}$ and $\{y_i\}_{i \in \mathbb{N}_t}$ as \mathbf{x}_t and \mathbf{y}_t , respectively, is rigorously equivalent to

$$\begin{aligned} \mathcal{P}(\mathcal{F}) &\equiv \int_{\mathcal{F}} \Lambda_t(\mathcal{X}_t(\omega), \mathcal{Y}_t(\omega)) d\tilde{\mathcal{P}}(\omega) \\ &\equiv \int_{\mathcal{B}} \Lambda_t(\mathbf{x}_t, \mathbf{y}_t) d^{2t} \tilde{\mathcal{P}}_{(\mathcal{X}_t, \mathcal{Y}_t)}(\mathbf{x}_t, \mathbf{y}_t) \\ &\equiv \mathcal{P}_{(\mathcal{X}_t, \mathcal{Y}_t)}(\mathcal{B}) \equiv \mathcal{P}((\mathcal{X}_t, \mathcal{Y}_t) \in \mathcal{B}), \end{aligned} \quad (21)$$

$$(22)$$

$$\forall \mathcal{F} \triangleq \{\omega \in \Omega \mid (\mathcal{X}_t(\omega), \mathcal{Y}_t(\omega)) \in \mathcal{B}\} \in \mathcal{H}_t \quad \text{and} \quad (23)$$

$$\forall \mathcal{B} \in \left(\otimes_{i \in \mathbb{N}_t} \mathcal{B}(\mathbb{R}^{N_i \times 1}) \right) \otimes \left(\otimes_{i \in \mathbb{N}_t} \mathcal{B}(\mathbb{R}^{M_i \times 1}) \right), \quad \forall t \in \mathbb{N}, \quad (24)$$

respectively (in the above, “ \otimes ” denotes the product operator for σ -algebras). Of course, the existence and almost everywhere uniqueness of Λ_t are guaranteed by the Radon-Nikodym Theorem, provided that the base measure \mathcal{P} is absolutely continuous with respect to $\tilde{\mathcal{P}}$ on \mathcal{H}_t ($\mathcal{P} \ll_{\mathcal{H}_t} \tilde{\mathcal{P}}$). Further, for the case where there exist densities characterizing \mathcal{P} and $\tilde{\mathcal{P}}$ (as in Theorem 1), demanding that $\mathcal{P} \ll_{\mathcal{H}_t} \tilde{\mathcal{P}}$ is precisely equivalent to demanding that (16) is true and, again through the Radon-Nikodym Theorem, it can be easily shown that the derivative Λ_t actually coincides with the likelihood ratio process defined in (17), almost everywhere. \blacksquare

Now, let us apply Theorem 1 for the stochastic processes X_t and \mathbf{y}_t , comprising our partially observed system, as defined in Section II.A. In this respect, we present the following result.

Theorem 2. (Change of Measure for the Hidden System under Study) *Consider the hidden stochastic system of Section II.A on the usual base space $(\Omega, \mathcal{F}, \mathcal{P})$, where $X_t \in \mathcal{Z}$ and $\mathbf{y}_t \in \mathbb{R}^{N \times 1}$, almost surely $\forall t \in \mathbb{N}$, constitute the hidden state process and the observation process, respectively. Then, there exists an alternative, **equivalent** to \mathcal{P} , base measure $\tilde{\mathcal{P}}$ on (Ω, \mathcal{F}) , under which:*

- *The processes X_t and \mathbf{y}_t are statistically independent.*
- *X_t constitutes a stochastic process with exactly the same dynamics as under \mathcal{P} .*
- *\mathbf{y}_t constitutes a Gaussian vector white noise process with zero mean and covariance matrix equal to the identity.*

Additionally, the filter \hat{X}_t can be expressed as in (18), where the $\{\mathcal{H}_t\}$ -adapted stochastic process $\Lambda_t, t \in \mathbb{N}$ is defined as in (25) (top of next page).

$$\begin{aligned}
\Lambda_t &\triangleq \prod_{i \in \mathbb{N}_t} \frac{\exp\left(\frac{1}{2} \|\mathbf{y}_i\|_2^2 - \frac{1}{2} (\mathbf{y}_i - \boldsymbol{\mu}_i(X_i))^T \left(\boldsymbol{\Sigma}_i(X_i) + \sigma_\xi^2 \mathbf{I}_{N \times N}\right)^{-1} (\mathbf{y}_i - \boldsymbol{\mu}_i(X_i))\right)}{\sqrt{\det\left(\boldsymbol{\Sigma}_i(X_i) + \sigma_\xi^2 \mathbf{I}_{N \times N}\right)}} \triangleq \prod_{i \in \mathbb{N}_t} \lambda_i \\
&\equiv \frac{\exp\left(\frac{1}{2} \sum_{i \in \mathbb{N}_t} \|\mathbf{y}_i\|_2^2 - (\mathbf{y}_i - \boldsymbol{\mu}_i(X_i))^T \left(\boldsymbol{\Sigma}_i(X_i) + \sigma_\xi^2 \mathbf{I}_{N \times N}\right)^{-1} (\mathbf{y}_i - \boldsymbol{\mu}_i(X_i))\right)}{\prod_{i \in \mathbb{N}_t} \sqrt{\det\left(\boldsymbol{\Sigma}_i(X_i) + \sigma_\xi^2 \mathbf{I}_{N \times N}\right)}} \in \mathbb{R}_{++} \quad (25)
\end{aligned}$$

Proof of Theorem 2. Additionally to the similar identifications made above (see (19)) and for later reference, let

$$\boldsymbol{\mathcal{Y}}_t \triangleq \{\mathbf{y}_i\}_{i \in \mathbb{N}_t} \quad \text{and} \quad \mathbf{y}_t \triangleq \{\mathbf{y}_i\}_{i \in \mathbb{N}_t}. \quad (26)$$

First, we construct the probability measure $\tilde{\mathcal{P}}_t$ this way showing its existence. To accomplish this, define, for each $t \in \mathbb{N}$, a probability measure $\tilde{\mathcal{P}}_{\mathcal{R}_t}$ on the measurable space $(\mathcal{R}_t, \mathcal{B}(\mathcal{R}_t))$, where

$$\mathcal{R}_t \triangleq \left(\prod_{i \in \mathbb{N}_t} \mathbb{R}\right) \times \left(\prod_{i \in \mathbb{N}_t} \mathbb{R}^{N \times 1}\right), \quad (27)$$

being absolutely continuous with respect to the Lebesgue measure on $(\mathcal{R}_t, \mathcal{B}(\mathcal{R}_t))$ and with density $\tilde{f}_t : \mathcal{R} \mapsto \mathbb{R}_+$. Since, for each $t \in \mathbb{N}$, the processes $X_t(\omega)$ and $\mathbf{y}_t(\omega)$ are both, by definition, *fixed* and *measurable* functions from (Ω, \mathcal{H}_t) to $(\mathcal{R}_t, \mathcal{B}(\mathcal{R}_t))$, with²

$$\mathcal{H}_t \subseteq \mathcal{H}_\infty \triangleq \sigma\left\{\bigcup_{t \in \mathbb{N}} \mathcal{H}_t\right\} \subseteq \mathcal{F}, \quad (28)$$

measuring any $\mathcal{B} \in \mathcal{B}(\mathcal{R}_t)$ under $\tilde{\mathcal{P}}_{\mathcal{R}_t}$ can be replaced by measuring the event (preimage) $\{\omega \in \Omega \mid (\mathcal{X}_t, \boldsymbol{\mathcal{Y}}_t) \in \mathcal{B}\} \in \mathcal{H}_t$ under another measure, say $\tilde{\mathcal{P}}$, defined collectively for all $t \in \mathbb{N}$ on the general measurable space $(\Omega, \mathcal{H}_\infty)$ as

$$\tilde{\mathcal{P}}(\{\omega \in \Omega \mid (\mathcal{X}_t, \boldsymbol{\mathcal{Y}}_t) \in \mathcal{B}\}) \equiv \tilde{\mathcal{P}}((\mathcal{X}_t, \boldsymbol{\mathcal{Y}}_t) \in \mathcal{B}) \triangleq \tilde{\mathcal{P}}_{\mathcal{R}_t}(\mathcal{B}), \quad \forall \mathcal{B} \in \mathcal{B}(\mathcal{R}_t).$$

That is, the restriction of the probability measure $\tilde{\mathcal{P}}$ to the σ -algebra \mathcal{H}_∞ is *induced* by the probability measure $\tilde{\mathcal{P}}_{\mathcal{R}_\infty}$ (also see Kolmogorov's Extension Theorem [3]). Further, in order to define the alternative base measure $\tilde{\mathcal{P}}$ fully on (Ω, \mathcal{F}) , we have to extend its behavior on the remaining events which belong to the potentially finer σ -algebra \mathcal{F} but are not included in \mathcal{H}_∞ . However, since we are interested in change of measure only for the augmented process $(\mathcal{X}_t, \boldsymbol{\mathcal{Y}}_t)$, these events are irrelevant to us. Therefore, $\tilde{\mathcal{P}}$ can be defined arbitrarily on these events, as long as it remains a valid and consistent probability measure.

² \mathcal{H}_∞ constitutes the *join*, that is, the *smallest* σ -algebra generated by the union of all $\mathcal{H}_t, \forall t \in \mathbb{N}$.

Now, to finalize the construction of the restriction of $\tilde{\mathcal{P}}$ to $\mathcal{H}_t, \forall t \in \mathbb{N}$, we have to explicitly specify the density of $\tilde{\mathcal{P}}_{\mathcal{R}_t}$, or, equivalently, of the joint density of the random variables $(\mathcal{X}_t, \mathcal{Y}_t)$, \tilde{f}_t , for all $t \in \mathbb{N}$. According to the statement of Theorem 2, we have to demand that

$$\begin{aligned}
\tilde{f}_t(\mathcal{X}_t, \mathcal{Y}_t) &\equiv \tilde{f}_{\mathcal{Y}_t|\mathcal{X}_t}(\mathcal{Y}_t|\mathcal{X}_t) \tilde{f}_{\mathcal{X}_t}(\mathcal{X}_t) \\
&= \tilde{f}_{\mathcal{Y}_t}(\mathcal{Y}_t) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&= \left(\prod_{i \in \mathbb{N}_t} \tilde{f}_{\mathcal{Y}_i}(\mathcal{Y}_i) \right) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&= \left(\prod_{i \in \mathbb{N}_t} \frac{\exp\left(-\frac{\|\mathcal{Y}_i\|_2^2}{2}\right)}{\sqrt{(2\pi)^N}} \right) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&\equiv \frac{\exp\left(-\frac{1}{2} \sum_{i \in \mathbb{N}_t} \|\mathcal{Y}_i\|_2^2\right)}{\sqrt{(2\pi)^{N(t+1)}}} f_{\mathcal{X}_t}(\mathcal{X}_t). \tag{29}
\end{aligned}$$

Next, by definition, we know that, under \mathcal{P} , the joint density of $(\mathcal{X}_t, \mathcal{Y}_t)$ can be expressed as

$$\begin{aligned}
f_t(\mathcal{X}_t, \mathcal{Y}_t) &\equiv f_{\mathcal{Y}_t|\mathcal{X}_t}(\mathcal{Y}_t|\mathcal{X}_t) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&\equiv \left(\prod_{i \in \mathbb{N}_t} f_{\mathcal{Y}_i|X_i}(\mathcal{Y}_i|x_i) \right) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&= \left(\prod_{i \in \mathbb{N}_t} \frac{\exp\left(\frac{\bar{\mathcal{Y}}_i^T \mathbf{C}_i^{-1} \bar{\mathcal{Y}}_i}{-2}\right)}{\sqrt{\det(\mathbf{C}_i)} (2\pi)^N} \right) f_{\mathcal{X}_t}(\mathcal{X}_t) \\
&\equiv \frac{\exp\left(\sum_{i \in \mathbb{N}_t} \frac{\bar{\mathcal{Y}}_i^T \mathbf{C}_i^{-1} \bar{\mathcal{Y}}_i}{-2}\right)}{\left(\prod_{i \in \mathbb{N}_t} \sqrt{\det(\mathbf{C}_i)} \right) \sqrt{(2\pi)^{N(t+1)}}} f_{\mathcal{X}_t}(\mathcal{X}_t), \tag{30}
\end{aligned}$$

where, for all $t \in \mathbb{N}$,

$$\bar{\mathcal{Y}}_t \equiv \bar{\mathcal{Y}}_t(x_t) \triangleq \mathcal{Y}_t - \boldsymbol{\mu}_t(x_t) \in \mathbb{R}^{N \times 1} \quad \text{and} \tag{31}$$

$$\mathbf{C}_t \equiv \mathbf{C}_t(x_t) \equiv \boldsymbol{\Sigma}_t(x_t) + \sigma_\xi^2 \mathbf{I}_{N \times N} \in \mathcal{D}_{\mathbf{C}}, \tag{32}$$

where $\mathcal{D}_{\mathbf{C}}$ constitutes a bounded subset of $\mathbb{R}^{N \times N}$. From (29) and (30), it is obvious that the sufficient condition (16) of Theorem 1 is satisfied (actually, in this case, we have an equivalence; as

a result, the change of measure is an invertible transformation). Applying Theorem 1, (18) must be true by defining the $\{\mathcal{H}_t\}$ -adapted stochastic process

$$\begin{aligned} \Lambda_t \equiv \Lambda_t(\mathcal{X}_t, \mathcal{Y}_t) &\triangleq \frac{f_t(\mathcal{X}_t, \mathcal{Y}_t)}{\tilde{f}_t(\mathcal{X}_t, \mathcal{Y}_t)} \equiv \frac{f_{\mathcal{Y}_t|\mathcal{X}_t}(\mathcal{Y}_t|\mathcal{X}_t)}{\tilde{f}_{\mathcal{Y}_t|\mathcal{X}_t}(\mathcal{Y}_t|\mathcal{X}_t)} \equiv \frac{f_{\mathcal{Y}_t|\mathcal{X}_t}(\mathcal{Y}_t|\mathcal{X}_t)}{\tilde{f}_{\mathcal{Y}_t}(\mathcal{Y}_t)} \\ &\equiv \frac{\exp\left(\sum_{i \in \mathbb{N}_t} \frac{\|\mathbf{y}_i\|_2^2 - (\bar{\mathbf{y}}_i(X_i))^T (\mathbf{C}_i(X_i))^{-1} \bar{\mathbf{y}}_i(X_i)}{2}\right)}{\prod_{i \in \mathbb{N}_t} \sqrt{\det(\mathbf{C}_i(X_i))}}, \end{aligned} \quad (33)$$

or, alternatively,

$$\Lambda_t \equiv \prod_{i \in \mathbb{N}_t} \lambda_i \triangleq \prod_{i \in \mathbb{N}_t} \frac{\exp\left(\sum_{i \in \mathbb{N}_t} \frac{\|\mathbf{y}_i\|_2^2 - (\bar{\mathbf{y}}_i(X_i))^T (\mathbf{C}_i(X_i))^{-1} \bar{\mathbf{y}}_i(X_i)}{2}\right)}{\sqrt{\det(\mathbf{C}_i(X_i))}}, \quad (34)$$

therefore completing the proof. ■

2.3 Weak & \mathcal{C} -Weak Convergence of (Random) Probability Measures

In the analysis that will take place in Section IV, we will make use of the notions of weak and conditionally weak (\mathcal{C} -weak) convergence of sequences of probability measures. Thus, let us define these notions of stochastic convergence consistently, suited at least for the purposes of our investigation.

Definition 1. (Weak Convergence [24]) Let \mathcal{S} be an arbitrary metric space, let $\mathcal{S} \triangleq \mathcal{B}(\mathcal{S})$ be the associated Borel σ -algebra and consider a sequence of probability measures $\{\pi_n\}_{n \in \mathbb{N}}$ on \mathcal{S} . If π constitutes another “limit” probability measure on \mathcal{S} such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \pi_n(\mathcal{A}) &= \pi(\mathcal{A}), \\ \forall \mathcal{A} \in \mathcal{S} \text{ such that } \pi(\partial \mathcal{A}) &\equiv 0, \end{aligned} \quad (35)$$

where $\partial \mathcal{A}$ denotes the boundary set of the Borel set \mathcal{A} , then we say that the sequence $\{\pi_n\}_{n \in \mathbb{N}}$ converges to π *weakly* or *in the weak sense* and we equivalently write

$$\pi_n \xrightarrow[n \rightarrow \infty]{\mathcal{W}} \pi. \quad (36)$$

Of course, weak convergence of probability measures is equivalent to weak convergence or convergence in distribution, in case we are given sequences of $(\mathcal{S}, \mathcal{S})$ -valued random variables whose induced probability measures converge in the aforementioned sense.

Next, we present a definition for conditionally weak convergence of probability measures. To avoid possibly complicating technicalities, this definition is not presented in full generality. Rather, it is presented in an appropriately specialized form, which will be used later on, in the analysis that follows.

Definition 2. (Conditionally Weak Convergence) Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a base probability triplet and consider the measurable spaces $(\mathcal{S}_i, \mathcal{S}_i \triangleq \mathcal{B}(\mathcal{S}_i))$, $i = \{1, 2\}$, where \mathcal{S}_1 and \mathcal{S}_2 constitute a complete separable metric (Polish) space and an arbitrary metric space, respectively. Also, let $\{X_1^n : \Omega \rightarrow \mathcal{S}_1\}_{n \in \mathbb{N}}$ be a sequence of random variables, let $X_2 : \Omega \rightarrow \mathcal{S}_2$ be another random variable and consider the sequence of (regular) induced conditional probability distributions (or measures) $\mathcal{P}_{X_1^n | X_2}^n : \mathcal{S}_1 \times \Omega \rightarrow [0, 1]$, such that

$$\mathcal{P}_{X_1^n | X_2}^n (\mathcal{A} | X_2(\omega)) \equiv \mathcal{P}(X_1^n \in \mathcal{A} | \sigma\{X_2\}), \quad (37)$$

\mathcal{P} - a.e., for any Borel set $\mathcal{A} \in \mathcal{S}_1$. If $X_1 : \Omega \rightarrow \mathcal{S}_1$ constitutes a “limit” random variable, whose induced conditional measure $\mathcal{P}_{X_1 | X_2} : \mathcal{S}_1 \times \Omega \rightarrow [0, 1]$ is such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{P}_{X_1^n | X_2}^n (\mathcal{A} | X_2(\omega)) &= \mathcal{P}_{X_1 | X_2} (\mathcal{A} | X_2(\omega)), \\ \forall \mathcal{A} \in \mathcal{S}_1 \text{ such that } \pi(\partial \mathcal{A}) &\equiv 0 \text{ and } \mathcal{P} - a.e., \end{aligned} \quad (38)$$

then we say that the sequence $\{\mathcal{P}_{X_1^n | X_2}^n\}_{n \in \mathbb{N}}$ converges to $\mathcal{P}_{X_1 | X_2}$ *conditionally weakly* (*C-weakly*) or *in the conditionally weak* (*C-weak*) *sense* and we equivalently write

$$\mathcal{P}_{X_1^n | X_2}^n (\cdot | X_2) \xrightarrow[n \rightarrow \infty]{\mathcal{W}} \mathcal{P}_{X_1 | X_2} (\cdot | X_2). \quad (39)$$

Remark 3. Actually, C-weak convergence, as defined above, is strongly related to the more general concepts of *almost sure weak convergence* and random probability measures. For instance, the reader is referred to the related articles [25] and [26]. ■

Further, the following lemma characterizes weak convergence of probability measures (and random variables) [24].

Lemma 1. (Weak Convergence & Expectations) *Let \mathcal{S} be an arbitrary metric space and let $\mathcal{S} \triangleq \mathcal{B}(\mathcal{S})$. Suppose we are given a sequence of random variables $\{X^n\}_{n \in \mathbb{N}}$ and a “limit” X , all $(\mathcal{S}, \mathcal{S})$ -valued, but possibly defined on different base probability spaces, with $\{\mathcal{P}_{X^n}\}_{n \in \mathbb{N}}$ and \mathcal{P}_X being their induced probability measures on \mathcal{S} , respectively. Then,*

$$X^n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} X \Leftrightarrow \mathcal{P}_{X^n} \xrightarrow[n \rightarrow \infty]{\mathcal{W}} \mathcal{P}_X, \quad (40)$$

if and only if

$$\mathbb{E}\{f(X^n)\} \equiv \int_{\mathcal{S}} f d\mathcal{P}_{X^n} \xrightarrow[n \rightarrow \infty]{} \int_{\mathcal{S}} f d\mathcal{P}_X \equiv \mathbb{E}\{f(X)\}, \quad (41)$$

for all bounded, continuous functions $f : \mathcal{S} \rightarrow \mathbb{R}$.

Of course, if we replace weak convergence by C-weak convergence, Lemma 1 continues to hold, but, in this case, (41) should be understood in the almost everywhere sense (see, for example, [26]). More specifically, under the generic notation of Definition 2 and under the appropriate assumptions according to Lemma 1, it will be true that

$$\mathbb{E}\{f(X_1^n) | X_2\}(\omega) \xrightarrow[n \rightarrow \infty]{} \mathbb{E}\{f(X_1) | X_2\}(\omega), \quad (42)$$

for almost all $\omega \in \Omega$.

3 Problem Formulation & Statement of Main Results

In this section, we formulate the problem of interest, that is, in a nutshell, the problem of approximating a nonlinear MMSE filter by another (asymptotically optimal) filtering operator, defined by replacing the true process we would like to filter by an appropriate approximation. Although we do not deal with such a problem here, such an approximation would be chosen in order to yield a practically realizable approximate filtering scheme. We also present the main result of the paper, establishing sufficient conditions for convergence of the respective approximate filters, in an indeed strong sense.

Let us start from the beginning. From Theorem 2, we know that

$$\mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\} \equiv \frac{\mathbb{E}_{\tilde{\mathcal{P}}} \{X_t \Lambda_t | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}} \{\Lambda_t | \mathcal{Y}_t\}}, \quad \forall t \in \mathbb{N}, \quad (43)$$

where the RHS constitutes an alternative representation for the filter on the LHS, which constitutes the optimal in the MMSE sense estimator of the partially observed process X_t , given the available observations up to time t . If the numerical evaluation of either of the sides of (43) is difficult (either we are interested in a recursive realization of the filter or not), one could focus on the RHS, where the state and the observations constitute independent processes, and, keeping the same observations, replace X_t by another process X_t^A , called the *approximation*, with *resolution* or *approximation parameter* $A \in \mathbb{N}$ (for simplicity), also independent of the observations (with respect to $\tilde{\mathcal{P}}$), for which the evaluation of the resulting “filter” might be easier. Under some appropriate, well defined sense, the approximation to the original process improves as $A \rightarrow \infty$. This general idea of replacing the true state process with an approximation is employed in, for instance, [10, 11], and will be employed here, too.

At this point, a natural question arises: Why are we complicating things with change of measure arguments and not using X_t^A directly in the LHS of (43)? Indeed, using classical results such as the Dominated Convergence Theorem, one could prove at least pointwise convergence of the respective filter approximations. The main and most important issue with such an approach is that, in order for such a filter to be realizable in any way, special attention must be paid to the choice of the approximation, regarding its stochastic dependence on the observations process. This is due to the original stochastic coupling between the state and the observations of the hidden system of interest. However, using change of measure, one can find an alternative representation of the filter process, where, under another probability measure, the state and observations are stochastically decoupled (independent). This makes the problem much easier, because the approximation can also be chosen to be independent of the observations. If we especially restrict our attention to recursive nonlinear filters, change of measure provides a rather versatile means for discovering recursive filter realizations. See, for example, the detailed treatment presented in [3].

Thus, concentrating on the RHS of (43), we can define an *approximate filtering operator* of the process X_t , also with resolution $A \in \mathbb{N}$, as

$$\mathcal{E}^A (X_t | \mathcal{Y}_t) \triangleq \frac{\mathbb{E}_{\tilde{\mathcal{P}}} \{X_t^A \Lambda_t^A | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}} \{\Lambda_t^A | \mathcal{Y}_t\}}, \quad \forall t \in \mathbb{N}. \quad (44)$$

Observe that the above quantity *is not a conditional expectation of X_t^A* , because X_t^A does not follow the probability law of the true process of interest, X_t [11]. Of course, the question is if and under

which sense,

$$\mathcal{E}^A (X_t | \mathcal{Y}_t) \xrightarrow[A \rightarrow \infty]{?} \mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\}, \quad (45)$$

that is, if and in which sense our chosen approximate filtering operator is *asymptotically optimal*, as the resolution of the approximation increases. In other words, we are looking for a class of approximations, whose members approximate the process X_t well, in the sense that the resulting approximate filtering operators converge to the true filter as the resolution parameter increases, that is, as $A \rightarrow \infty$, and under some appropriate notion of convergence. In this respect, below we formulate and prove the following theorem, which constitutes the main result of this paper (recall the definition of \mathcal{C} -weak convergence given in Section II.C). In the following, $\mathbb{1}_{\mathcal{A}} : \mathbb{R} \rightarrow \{0, 1\}$ denotes the indicator of the set \mathcal{A} . Also, for any Borel set \mathcal{A} , $\mathbb{1}_{\mathcal{A}}(\cdot)$ constitutes a Dirac (atomic) probability measure. Equivalently, we write $\mathbb{1}_{\mathcal{A}}(\cdot) \equiv \delta_{(\cdot)}(\mathcal{A})$.

Theorem 3. (Convergence to the Optimal Filter) *Pick any natural $T < \infty$ and suppose either of the following:*

- For all $t \in \mathbb{N}_T$, the sequence $\left\{X_t^A\right\}_{A \in \mathbb{N}}$ is marginally \mathcal{C} -weakly convergent to X_t , given X_t , that is,

$$\mathcal{P}_{X_t^A | X_t}(\cdot | X_t) \xrightarrow[A \rightarrow \infty]{\mathcal{W}} \delta_{X_t}(\cdot), \quad \forall t \in \mathbb{N}_T. \quad (46)$$

- For all $t \in \mathbb{N}_T$, the sequence $\left\{X_t^A\right\}_{A \in \mathbb{N}}$ is (marginally) convergent to X_t in probability, that is,

$$X_t^A \xrightarrow[A \rightarrow \infty]{\mathcal{P}} X_t, \quad \forall t \in \mathbb{N}_T. \quad (47)$$

Then, there exists a measurable subset $\widehat{\Omega}_T \subseteq \Omega$ with \mathcal{P} -measure at least $1 - (T + 1)^{1-CN} \exp(-CN)$, such that

$$\lim_{A \rightarrow \infty} \sup_{t \in \mathbb{N}_T} \sup_{\omega \in \widehat{\Omega}_T} \left| \mathcal{E}^A (X_t | \mathcal{Y}_t) - \mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\} \right|(\omega) \equiv 0, \quad (48)$$

for any free, finite constant $C \geq 1$. In other words, the convergence of the respective approximate filtering operators is compact in $t \in \mathbb{N}$ and, with probability at least $1 - (T + 1)^{1-CN} \exp(-CN)$, uniform in ω .

Interestingly, as noted in the beginning of this section, the mode of convergence of the resulting approximate filtering operator is particularly strong. In fact, it is interesting that, for fixed T , the approximate filter $\mathcal{E}^A (X_t | \mathcal{Y}_t)$ converges to $\mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\}$ (uniformly) in a set that approaches the certain event, exponentially in N . That is, convergence to the optimal filter *tends to be in the uniformly almost everywhere sense, at an exponential rate (in N)*. Consequently, it is revealed that the dimensionality of the observations process essentially stabilizes the behavior of the approximate filter, in a stochastic sense. Along the lines of the discussion presented above, it is clear that Theorem 3 provides a way of quantitatively justifying Egoroff's theorem [12], which bridges almost uniform convergence with almost sure convergence, however in an indeed abstract fashion.

Remark 4. The \mathcal{C} -weak convergence condition (46) is a rather strong one. In particular, as we show later in Lemma 8 (see Section IV), it implies \mathcal{L}_1 convergence, which means that it also implies (marginal) convergence in probability (which constitutes the alternative sufficient condition of Theorem 3). In simple words, (46) resembles a situation where, at any time step, one is given

or defines an approximation to the original process, in the sense that, *conditioned on the original process* at the same time step, the probability of being equal to the latter approaches unity. At this point, because \mathcal{C} -weak convergence is stronger than (and implies) convergence in probability, one could wonder why we presented both as alternative sufficient conditions for filter convergence in Theorem 3 (and also in Lemma 10 presented in Section IV). The reason is that, contrary to convergence in probability, condition (46) provides a nice *structural* criterion for *constructing* state process approximations in a natural way, which is also consistent with our intuition: If, at any time step, we could observe the value of true state process, then the respective value of the approximation at that same time step should be “sufficiently close” to the value of the state. Condition (46) expresses this intuitive idea and provides a version of the required sense of “closeness”. ■

In order to demonstrate the applicability of Theorem 3, as well as demystify the \mathcal{C} -weak convergence condition (46), let us present a simple but illustrative example. The example refers to a class of approximate grid based filters, based on the so called marginal approximation [14, 18], according to which the (compactly restricted) state process is fed into a uniform spatial quantizer of variable resolution. As we will see, this intuitively reasonable approximation idea constitutes a simple instance of the condition (95).

More specifically, assume that $X_t \in [a, b] \equiv \mathcal{Z}$, $\forall t \in \mathbb{N}$, almost surely. Let us discretize \mathcal{Z} uniformly into \mathbf{A} subintervals, of identical length, called cells. The l -th cell and its respective center are denoted as \mathcal{Z}_A^l and x_A^l , $l \in \mathbb{N}_A^+$. Then, letting $\mathcal{X}_A \triangleq \left\{ x_A^l \right\}_{l \in \mathbb{N}_A^+}$, the *quantizer* $\mathcal{Q}_A : (\mathcal{Z}, \mathcal{B}(\mathcal{Z})) \mapsto (\mathcal{X}_A, 2^{\mathcal{X}_A})$ is defined as the bijective and measurable function which uniquely maps the l -th cell to the respective *reconstruction point* x_A^l , $\forall l \in \mathbb{N}_A^+$. That is, $\mathcal{Q}_A(x) \triangleq x_A^l$ if and only if $x \in \mathcal{Z}_A^l$ [14]. Having defined the quantizer $\mathcal{Q}_A(\cdot)$, the *Marginal Quantization* of the state is defined as [18]

$$X_t^A(\omega) \triangleq \mathcal{Q}_A(X_t(\omega)) \in \mathcal{X}_A, \forall t \in \mathbb{N}, \mathcal{P} - a.s., \quad (49)$$

where $\mathbf{A} \in \mathbb{N}$ is identified as the approximation parameter. That is, X_t is approximated by its nearest neighbor on the cell grid. That is, the state is represented by a discrete set of reconstruction points, each one of them uniquely corresponding to a member of a partition of \mathcal{Z} .

By construction of marginal state approximations, it can be easily shown that [14]

$$X_t^A(\omega) \xrightarrow[\mathbf{A} \rightarrow \infty]{\mathcal{P}-a.s.} X_t(\omega), \quad (50)$$

a fact that will be used in the following. Of course, almost sure convergence implies convergence in probability and, as we will see, \mathcal{C} -weak convergence as well. First, let us determine the conditional probability measure $\mathcal{P}_{X_t^A|X_t}^A(dx|X_t)$. Since knowing X_t uniquely determines the value of X_t^A , it must be true that

$$\begin{aligned} \mathcal{P}_{X_t^A|X_t}^A(dx|X_t) &\equiv \mathcal{P}_{\mathcal{Q}_A(X_t)|X_t}^A(dx|X_t) \\ &\equiv \delta_{\mathcal{Q}_A(X_t)}(dx), \quad \mathcal{P} - a.s.. \end{aligned} \quad (51)$$

However, from Lemma 1, we know that weak convergence of measures is equivalent to showing that the expectations $\mathbb{E} \left\{ f \left(X_t^A \right) \middle| X_t \right\}$ converge to $\mathbb{E} \{ f(X_t) | X_t \} \equiv f(X_t)$, for all bounded and continuous $f(\cdot)$, almost everywhere. Indeed,

$$\mathbb{E} \left\{ f \left(X_t^A \right) \middle| X_t \right\}(\omega) \equiv \int_{\mathcal{Z}} f(x) \mathcal{P}_{X_t^A|X_t}^A(dx|X_t(\omega))$$

$$\begin{aligned}
&\equiv \int_{\mathcal{Z}} f(x) \delta_{\mathcal{Q}_A(X_t(\omega))}(\mathrm{d}x) \\
&\equiv f(\mathcal{Q}_A(X_t(\omega))) \xrightarrow[\mathbf{A} \rightarrow \infty]{\mathcal{P}\text{-a.s.}} f(X_t(\omega)),
\end{aligned} \tag{52}$$

due to the continuity of $f(\cdot)$. Consequently, we have shown that

$$\mathcal{P}_{X_t^A | X_t}^A(\cdot | X_t) \equiv \delta_{\mathcal{Q}_A(X_t)}(\cdot) \xrightarrow[\mathbf{A} \rightarrow \infty]{\mathcal{W}} \delta_{X_t}(\cdot), \tag{53}$$

fulfilling the first requirement of Theorem 3. This very simple example constitutes the basis for constructing more complicated and cleverly designed state approximations (for example, using stochastic quantizers). The challenge here is to come up with such approximations exhibiting nice properties, which would potentially lead to the development of effective approximate recursive or, in general, sequential filtering schemes, well suited for dynamic inference in complex partially observable stochastic nonlinear systems. As far as grid based approximate recursive filtering is concerned, a relatively complete discussion of the problem is presented in the recent paper [14], where marginal state approximations are also treated in full generality.

An important and direct consequence of Theorem 3, also highlighted by the example presented above, is that, interestingly, the nature of the state process is completely irrelevant when one is interested in convergence of the respective approximate filters, in the respective sense of the aforementioned theorem. This fact has the following pleasing and intuitive interpretation: It implies that if any of the two conditions of Theorem 3 are satisfied, then we should forget about the internal stochastic structure of the state, and instead focus exclusively on the way the latter is being observed through time. That is, *we do not really care about what we partially observe, but how well we observe it; and if we observe it well, we can filter it well, too*. Essentially, the observations should constitute a stable functional of the state, of course in some well defined sense. In this work, this notion of stability is expressed precisely through Assumption 1 and 2, presented earlier in Section II.

Note, however, that the existence of a consistent approximate filter in the sense of Theorem 3 does not automatically imply that this filter will be efficiently implementable; usually, we would like such a filter to admit a recursive/sequential representation (or possibly a semirecursive one [20]). As it turns out, this can happen when the chosen state approximation admits a valid semimartingale type representation (in addition to satisfying one of the sufficient conditions of Theorem 3). For example, the case where the state is Markovian and the chosen state approximation is of the marginal type, discussed in the basic example presented above, is treated in detail in [14].

Remark 5. The filter representation (43) coincides with the respective expression employed in importance sampling [19, 27]. Since, under the alternative measure $\tilde{\mathcal{P}}$, the observations and state constitute statistically independent processes, one can directly sample from the (joint) distribution of the state, fixing the observations to their respective value at each time t (of course, assuming that a relevant “sampling device” exists). However, note that that due to the assumptions of Theorem 3, related at least to convergence in probability of the corresponding state approximations, the aforementioned result cannot be used directly in order to show convergence of importance sampling or related particle filtering techniques, which are directly related to empirical measures. The possible ways Theorem 3 can be utilized in order to provide asymptotic guarantees for particle filtering (using additional assumptions) constitutes an interesting open topic for further research. ■

The rest of the paper is fully devoted in the detailed proof of Theorem 3.

4 Proof of Theorem 3

In order to facilitate the presentation, the proof is divided in a number of subsections.

4.1 Two Basic Lemmata, Linear Algebra - Oriented

Parts of the following useful results will be employed several times in the analysis that follows³.

Lemma 2. *Consider arbitrary matrices $\mathbf{A} \in \mathbb{C}^{N_1 \times M_1}$, $\mathbf{B} \in \mathbb{C}^{N_1 \times M_1}$, $\mathbf{X} \in \mathbb{C}^{M_2 \times N_2}$, $\mathbf{Y} \in \mathbb{C}^{M_2 \times N_2}$, and let $\|\cdot\|_{\mathfrak{M}}$ be any matrix norm. Then, the following hold:*

- *If either*
 - $N_1 \equiv M_1 \equiv 1$, *or*
 - $N_1 \equiv N_2 \equiv M_1 \equiv M_2$ *and* $\|\cdot\|_{\mathfrak{M}}$ *is submultiplicative,*

then

$$\|\mathbf{AX} - \mathbf{BY}\|_{\mathfrak{M}} \leq \|\mathbf{A}\|_{\mathfrak{M}} \|\mathbf{X} - \mathbf{Y}\|_{\mathfrak{M}} + \|\mathbf{Y}\|_{\mathfrak{M}} \|\mathbf{A} - \mathbf{B}\|_{\mathfrak{M}}. \quad (54)$$

- *If $N_2 \equiv 1$, $M_1 \equiv M_2$ and $\|\cdot\|_{\mathfrak{M}}$ constitutes any subordinate matrix norm to the ℓ_p vector norm, $\|\cdot\|_p$, then*

$$\|\mathbf{AX} - \mathbf{BY}\|_p \leq \|\mathbf{A}\|_{\mathfrak{M}} \|\mathbf{X} - \mathbf{Y}\|_p + \|\mathbf{Y}\|_p \|\mathbf{A} - \mathbf{B}\|_{\mathfrak{M}}. \quad (55)$$

Proof of Lemma 2. We prove the result only for the case where $N_1 \equiv N_2 \equiv M_1 \equiv M_2$ and $\|\cdot\|_{\mathfrak{M}}$ is submultiplicative. By definition of such a matrix norm,

$$\begin{aligned} \|\mathbf{AX} - \mathbf{BY}\|_{\mathfrak{M}} &\equiv \|\mathbf{AX} + \mathbf{AY} - \mathbf{AY} - \mathbf{BY}\|_{\mathfrak{M}} \\ &\equiv \|\mathbf{A}(\mathbf{X} - \mathbf{Y}) + (\mathbf{A} - \mathbf{B})\mathbf{Y}\|_{\mathfrak{M}} \\ &\leq \|\mathbf{A}(\mathbf{X} - \mathbf{Y})\|_{\mathfrak{M}} + \|(\mathbf{A} - \mathbf{B})\mathbf{Y}\|_{\mathfrak{M}} \\ &\leq \|\mathbf{A}\|_{\mathfrak{M}} \|\mathbf{X} - \mathbf{Y}\|_{\mathfrak{M}} + \|\mathbf{Y}\|_{\mathfrak{M}} \|(\mathbf{A} - \mathbf{B})\|_{\mathfrak{M}}, \end{aligned} \quad (56)$$

apparently completing the proof. The results for the other two cases considered in Lemma 2 can be readily shown following similar procedure. ■

Lemma 3. *Consider the collections of arbitrary, square matrices*

$$\left\{ \mathbf{A}_i \in \mathbb{C}^{N \times N} \right\}_{i \in \mathbb{N}_n} \quad \text{and} \quad \left\{ \mathbf{B}_i \in \mathbb{C}^{N \times N} \right\}_{i \in \mathbb{N}_n}.$$

Then, for any submultiplicative matrix norm $\|\cdot\|_{\mathfrak{M}}$, it is true that

$$\left\| \prod_{i=0}^n \mathbf{A}_i - \prod_{i=0}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \leq \sum_{i=0}^n \left(\prod_{j=0}^{i-1} \|\mathbf{A}_j\|_{\mathfrak{M}} \right) \left(\prod_{j=i+1}^n \|\mathbf{B}_j\|_{\mathfrak{M}} \right) \|\mathbf{A}_i - \mathbf{B}_i\|_{\mathfrak{M}}. \quad (57)$$

³In this paper, Lemma 3 presented in this subsection will be applied only for scalars (and where the metric considered coincides with the absolute value). However, the general version of the result (considering matrices and submultiplicative norms) is presented for the sake of generality.

Proof of Lemma 3. Applying Lemma 2 to the LHS of (57), we get

$$\begin{aligned} \left\| \prod_{i=0}^n \mathbf{A}_i - \prod_{i=0}^n \mathbf{B}_i \right\|_{\mathfrak{M}} &\equiv \left\| \mathbf{A}_0 \prod_{i=1}^n \mathbf{A}_i - \mathbf{B}_0 \prod_{i=1}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \\ &\leq \|\mathbf{A}_0\|_{\mathfrak{M}} \left\| \prod_{i=1}^n \mathbf{A}_i - \prod_{i=1}^n \mathbf{B}_i \right\|_{\mathfrak{M}} + \left\| \prod_{i=1}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \|\mathbf{A}_0 - \mathbf{B}_0\|_{\mathfrak{M}}. \end{aligned} \quad (58)$$

The repeated application of Lemma 2 to the quantity multiplying $\|\mathbf{A}_0\|_{\mathfrak{M}}$ on the RHS of the expression above yields

$$\begin{aligned} \left\| \prod_{i=0}^n \mathbf{A}_i - \prod_{i=0}^n \mathbf{B}_i \right\|_{\mathfrak{M}} &\leq \|\mathbf{A}_0\|_{\mathfrak{M}} \|\mathbf{A}_1\|_{\mathfrak{M}} \left\| \prod_{i=2}^n \mathbf{A}_i - \prod_{i=2}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \\ &\quad + \|\mathbf{A}_0\|_{\mathfrak{M}} \left\| \prod_{i=2}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \|\mathbf{A}_1 - \mathbf{B}_1\|_{\mathfrak{M}} + \left\| \prod_{i=1}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \|\mathbf{A}_0 - \mathbf{B}_0\|_{\mathfrak{M}}, \end{aligned} \quad (59)$$

where, the ‘‘temporal pattern’’ is apparent. Indeed, iterating (59) and proceeding inductively, we end up with the bound

$$\left\| \prod_{i=0}^n \mathbf{A}_i - \prod_{i=0}^n \mathbf{B}_i \right\|_{\mathfrak{M}} \leq \sum_{i=0}^n \left(\prod_{j=0}^{i-1} \|\mathbf{A}_j\|_{\mathfrak{M}} \right) \left\| \prod_{j=i+1}^n \mathbf{B}_j \right\|_{\mathfrak{M}} \|\mathbf{A}_i - \mathbf{B}_i\|_{\mathfrak{M}} \quad (60)$$

and the result readily follows invoking the submultiplicativeness of $\|\cdot\|_{\mathfrak{M}}$. \blacksquare

4.2 Preliminary Results

Here, we present and prove a number of preliminary results, which will help us towards the proof of an important lemma, which will be the key to showing the validity of Theorem 3.

First, under Assumption 2, stated in Section II.A, the following trivial lemmata hold.

Lemma 4. *Each member of the functional family $\{\Sigma_t : \mathcal{Z} \mapsto \mathcal{D}_{\Sigma}\}_{t \in \mathbb{N}}$ is Lipschitz continuous on \mathcal{Z} , in the Euclidean topology induced by the Frobenius norm. That is, $\forall t \in \mathbb{N}$,*

$$\|\Sigma_t(x) - \Sigma_t(y)\|_F \leq (NK_{\Sigma}) |x - y|, \quad (61)$$

$\forall (x, y) \in \mathcal{Z} \times \mathcal{Z}$, for the same constant $K_{\Sigma} \in \mathbb{R}_+$, as defined in Assumption 2. The same also holds for the family $\{\mathbf{C}_t : \mathcal{Z} \mapsto \mathcal{D}_{\mathbf{C}}\}_{t \in \mathbb{N}}$.

Proof of Lemma 4. By definition of the Frobenius norm,

$$\begin{aligned} \|\Sigma_t(x) - \Sigma_t(y)\|_F &\equiv \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} \left(\Sigma_t^{ij}(x) - \Sigma_t^{ij}(y) \right)^2} \\ &\leq \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} K_{\Sigma}^2 |x - y|^2} \\ &\equiv \sqrt{N^2 K_{\Sigma}^2 |x - y|^2}, \quad \forall t \in \mathbb{N} \end{aligned} \quad (62)$$

and our first claim follows. The second follows trivially if we recall the definition of each $\mathbf{C}_t(x)$. \blacksquare

Lemma 5. For each member of the functional family $\{\mathbf{C}_t : \mathcal{Z} \mapsto \mathcal{D}_{\mathbf{C}}\}_{t \in \mathbb{N}}$, it is true that, $\forall t \in \mathbb{N}$,

$$|\det(\mathbf{C}_t(x)) - \det(\mathbf{C}_t(y))| \leq (NK_{DET}) \|\mathbf{C}_t(x) - \mathbf{C}_t(y)\|_F, \quad (63)$$

$\forall (x, y) \in \mathcal{Z} \times \mathcal{Z}$, for some bounded constant $K_{DET} \equiv K_{DET}(N) \in \mathbb{R}_+$, possibly dependent on N but independent of t .

Proof of Lemma 5. As a consequence of the fact that the determinant of a matrix can be expressed as a polynomial function in N^2 variables (for example, see the Leibniz formula), it must be true that, $\forall t \in \mathbb{N}$,

$$\begin{aligned} |\det(\mathbf{C}_t(x)) - \det(\mathbf{C}_t(y))| &\leq K_{DET} \sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} \left| \mathbf{C}_t^{ij}(x) - \mathbf{C}_t^{ij}(y) \right| \\ &\equiv K_{DET} \|\mathbf{C}_t(x) - \mathbf{C}_t(y)\|_1, \end{aligned} \quad (64)$$

where the constant K_{DET} depends on *maximized* (using the fact that the domain $\mathcal{D}_{\mathbf{C}}$ is bounded) $(N-1)$ -fold products of elements of $\mathbf{C}_t(x)$ and $\mathbf{C}_t(y)$, with respect to x (resp. y) and t . Consequently, although K_{DET} may depend on N , it certainly does not depend on t . Now, since the ℓ_1 entrywise norm of an $N \times N$ matrix corresponds to the norm of a vector with N^2 elements, we may further bound the right hand side of the expression above by the Frobenius norm of $\mathbf{C}_t(x) - \mathbf{C}_t(y)$, yielding

$$|\det(\mathbf{C}_t(x)) - \det(\mathbf{C}_t(y))| \leq NK_{DET} \|\mathbf{C}_t(x) - \mathbf{C}_t(y)\|_F, \quad (65)$$

which is what we were set to prove. ■

Remark 6. The fact that the constant K_{DET} may be a function of the dimension of the observation vector, N , does not constitute a significant problem throughout our analysis, simply because N is always considered a *finite and fixed* parameter of our problem. However, it is true that the (functional) way N appears in the various constants in our derived expressions can potentially affect speed of convergence and, for that reason, it constitutes an important analytical aspect. Therefore, throughout the analysis presented below, a great effort has been made in order to keep the dependence of our bounds on N within reasonable limits. ■

We also present another useful lemma, related to the expansiveness of each member of the functional family $\{\mathbf{C}_t^{-1} : \mathcal{Z} \mapsto \mathcal{D}_{\mathbf{C}^{-1}}\}_{t \in \mathbb{N}}$.

Lemma 6. Each member of the functional family $\{\mathbf{C}_t^{-1} : \mathcal{Z} \mapsto \mathcal{D}_{\mathbf{C}^{-1}}\}_{t \in \mathbb{N}}$ is Lipschitz continuous on \mathcal{Z} , in the Euclidean topology induced by the Frobenius norm. That is, $\forall t \in \mathbb{N}$,

$$\left\| \mathbf{C}_t^{-1}(x) - \mathbf{C}_t^{-1}(y) \right\|_F \leq K_{INV} |x - y|, \quad (66)$$

$\forall (x, y) \in \mathcal{Z} \times \mathcal{Z}$, for some bounded constant $K_{INV} \equiv K_{INV}(N) \in \mathbb{R}_+$, possibly dependent on N but independent of t .

Proof of Lemma 6. As a consequence of Laplace's formula for the determinant of a matrix and invoking Lemma 2, it is true that

$$\left\| \mathbf{C}_t^{-1}(x) - \mathbf{C}_t^{-1}(y) \right\|_F$$

$$\begin{aligned}
&\equiv \left\| \frac{\text{adj}(\mathbf{C}_t(x))}{\det(\mathbf{C}_t(x))} - \frac{\text{adj}(\mathbf{C}_t(y))}{\det(\mathbf{C}_t(y))} \right\|_F \\
&\leq \frac{\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F}{\det(\mathbf{C}_t(x))} + \|\text{adj}(\mathbf{C}_t(y))\|_F \frac{|\det(\mathbf{C}_t(x)) - \det(\mathbf{C}_t(y))|}{\det(\mathbf{C}_t(x)) \det(\mathbf{C}_t(y))}, \tag{67}
\end{aligned}$$

where $\text{adj}(\mathbf{A})$ denotes the adjugate of the square matrix \mathbf{A} . Since $\mathbf{C}_t(x)$ (resp. $\mathbf{C}_t(y)$) is a symmetric and positive definite matrix, so is its adjugate. Employing one more property regarding the eigenvalues of the adjugate [28] and the fact that $\lambda_{inf} > 1$, we can write

$$\begin{aligned}
\|\text{adj}(\mathbf{C}_t(y))\|_F &\leq \sqrt{N} \|\text{adj}(\mathbf{C}_t(y))\|_2 \\
&\equiv \sqrt{N} \lambda_{max}(\text{adj}(\mathbf{C}_t(y))) \\
&\equiv \sqrt{N} \max_{i \in \mathbb{N}_N^+} \prod_{j \neq i} \lambda_j(\mathbf{C}_t(y)) \\
&\leq \sqrt{N} \det(\mathbf{C}_t(y)), \tag{68}
\end{aligned}$$

and then (67) becomes

$$\begin{aligned}
\left\| \mathbf{C}_t^{-1}(x) - \mathbf{C}_t^{-1}(y) \right\|_F &\leq \frac{\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F}{\det(\mathbf{C}_t(x))} + \sqrt{N} \frac{|\det(\mathbf{C}_t(x)) - \det(\mathbf{C}_t(y))|}{\det(\mathbf{C}_t(x))} \\
&\leq \frac{\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F}{\lambda_{inf}^N} + \frac{N^3 K_{DET} K_{\Sigma}}{\lambda_{inf}^N} |x - y|. \tag{69}
\end{aligned}$$

Next, the numerator of the first fraction from the left may be expressed as

$$\begin{aligned}
\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F &\equiv \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} \left(\text{adj}(\mathbf{C}_t(x))^{ij} - \text{adj}(\mathbf{C}_t(y))^{ij} \right)^2} \\
&\equiv \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} \left((-1)^{i+j} [\mathcal{M}_{ij}(\mathbf{C}_t(x)) - \mathcal{M}_{ij}(\mathbf{C}_t(y))] \right)^2} \\
&\equiv \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} \left(\mathcal{M}_{ij}(\mathbf{C}_t(x)) - \mathcal{M}_{ij}(\mathbf{C}_t(y)) \right)^2}, \tag{70}
\end{aligned}$$

where $\mathcal{M}_{ij}(\mathbf{C}_t(x))$ denotes the (i, j) -th minor of $\mathbf{C}_t(x)$, which constitutes the *determinant* of the $(N-1) \times (N-1)$ matrix formulated by removing the i -th row and the j -th column of $\mathbf{C}_t(x)$. Consequently, from Lemma 5, there exists a constant K_{det} , possibly dependent on N , such that, $\forall t \in \mathbb{N}$,

$$\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F \leq \sqrt{\sum_{(i,j) \in \mathbb{N}_N^+ \times \mathbb{N}_N^+} N^4 K_{det}^2 K_{\Sigma}^2 |x - y|^2}, \tag{71}$$

or, equivalently,

$$\|\text{adj}(\mathbf{C}_t(x)) - \text{adj}(\mathbf{C}_t(y))\|_F \leq N^3 K_{det} K_{\Sigma} |x - y|, \tag{72}$$

$\forall (x, y) \in \mathcal{Z} \times \mathcal{Z}$. Therefore, combining with (69), we get

$$\begin{aligned} \left\| \mathbf{C}_t^{-1}(x) - \mathbf{C}_t^{-1}(y) \right\|_F &\leq \frac{N^3}{\lambda_{inf}^N} (K_{DET} + K_{det}) K_{\Sigma} |x - y| \\ &\leq \frac{27\lambda_{inf}^{-3/\log(\lambda_{inf})}}{(\log(\lambda_{inf}))^3} (K_{DET} + K_{det}) K_{\Sigma} |x - y| \\ &\triangleq K_{INV} |x - y|, \end{aligned} \quad (73)$$

and the proof is complete. \blacksquare

Next, we state the following simple probabilistic result, related to the expansiveness of the norm of the observation vector in a stochastic sense, under both base measures \mathcal{P} and $\tilde{\mathcal{P}}$ considered throughout the paper (see Section II.B).

Lemma 7. *Consider the random quadratic form*

$$Q_t(\omega) \triangleq \|\mathbf{y}_t(\omega)\|_2^2 \equiv \|\bar{\mathbf{y}}_t(X_t(\omega)) + \boldsymbol{\mu}_t(X_t(\omega))\|_2^2, \quad t \in \mathbb{N}. \quad (74)$$

Then, for any fixed $t \in \mathbb{N}$ and any freely chosen $C \geq 1$, there exists a bounded constant $\gamma > 1$, such that the measurable set

$$\mathcal{T}_t \triangleq \left\{ \omega \in \Omega \left| \sup_{i \in \mathbb{N}_t} Q_i(\omega) < \gamma C N (1 + \log(t+1)) \right. \right\} \quad (75)$$

satisfies

$$\min \left\{ \mathcal{P}(\mathcal{T}_t), \tilde{\mathcal{P}}(\mathcal{T}_t) \right\} \geq 1 - \frac{\exp(-CN)}{(t+1)^{CN-1}}, \quad (76)$$

that is, the sequence of quadratic forms $\{Q_i(\omega)\}_{i \in \mathbb{N}_t}$ is uniformly bounded with very high probability under both base measures \mathcal{P} and $\tilde{\mathcal{P}}$.

Proof of Lemma 7. First, it is true that

$$\begin{aligned} \|\mathbf{y}_t(\omega)\|_2^2 &\equiv \|\bar{\mathbf{y}}_t(X_t(\omega)) + \boldsymbol{\mu}_t(X_t(\omega))\|_2^2 \\ &\equiv \|\bar{\mathbf{y}}_t(X_t(\omega))\|_2^2 + 2\mathbf{y}_t^T(X_t(\omega)) \boldsymbol{\mu}_t(X_t(\omega)) + \|\boldsymbol{\mu}_t(X_t(\omega))\|_2^2 \\ &\leq \|\bar{\mathbf{y}}_t(X_t(\omega))\|_2^2 + 2\|\bar{\mathbf{y}}_t(X_t(\omega))\|_2 \mu_{sup} + \mu_{sup}^2. \end{aligned} \quad (77)$$

Also, under \mathcal{P} , for each $t \in \mathbb{N}$, the random variable $\bar{\mathbf{y}}_t(X_t)$ constitutes an N -dimensional, conditionally (on X_t) Gaussian random variable with zero mean and covariance matrix $\mathbf{C}_t(X_t)$, that is

$$\bar{\mathbf{y}}_t | X_t \sim \mathcal{N}(0, \mathbf{C}_t(X_t) \equiv \mathbf{C}_{\bar{\mathbf{y}}_t | X_t}). \quad (78)$$

Then, if X_t is given,

$$\bar{Q}_t(\omega) \triangleq \|\bar{\mathbf{y}}_t(X_t(\omega))\|_2^2 \quad (79)$$

can be shown to admit the very useful alternative representation (for instance, see [29], pp. 89 - 90)

$$\bar{Q}_t \equiv \sum_{j \in \mathbb{N}_N^+} \lambda_j(\mathbf{C}_t(X_t)) U_j^2, \quad \forall t \in \mathbb{N}, \quad \text{with} \quad (80)$$

$$\{U_j\}_{j \in \mathbb{N}_N^+} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1). \quad (81)$$

From (80), one can readily observe that the statistical dependence of \bar{Q}_t on X_t concentrates only on the eigenvalues of the covariance matrix $\mathbf{C}_t(X_t)$, for which we have already assumed the existence of a finite supremum explicitly (see Assumption 1). Consequently, *conditioning on the process X_t* , we can bound (80) as

$$\bar{Q}_t \leq \lambda_{sup} \sum_{j \in \mathbb{N}_N^+} U_j^2 \triangleq \lambda_{sup} U, \quad \text{with } U \sim \chi^2(N), \quad (82)$$

almost everywhere and everywhere in time, where the RHS is independent of X_t . Next, from ([30], p. 1325), we know that for any chi squared random variable U with N degrees of freedom,

$$\mathcal{P}(U \geq N + 2\sqrt{Nu} + 2u) \leq \exp(-u), \quad \forall u > 0. \quad (83)$$

Setting $u \equiv CN(1 + \log(t+1))$ for any $C \geq 1$ and any $t \in \mathbb{N}$,

$$\mathcal{P}\left(U \geq N + 2N\sqrt{C(1 + \log(t+1))} + 2CN(1 + \log(t+1))\right) \leq \frac{\exp(-CN)}{(t+1)^{CN}}. \quad (84)$$

a statement which equivalently means that, with probability at least $1 - (t+1)^{-CN} \exp(-CN)$,

$$U < N + 2N\sqrt{C(1 + \log(t+1))} + 2CN(1 + \log(t+1)). \quad (85)$$

However, because the RHS of the above inequality is upper bounded by $5CN(1 + \log(t+1))$,

$$\begin{aligned} & \mathcal{P}(U < 5CN(1 + \log(t+1))) \\ & \geq \mathcal{P}\left(U < N + 2N\sqrt{C(1 + \log(t+1))} + 2CN(1 + \log(t+1))\right) \geq 1 - \frac{\exp(-CN)}{(t+1)^{CN}}. \end{aligned} \quad (86)$$

Hence, $\forall i \in \mathbb{N}_t$,

$$\begin{aligned} \mathcal{P}(\bar{Q}_i \geq 5\lambda_{sup}CN(1 + \log(t+1)) | X_i) & \leq \mathcal{P}(U \geq 5CN(1 + \log(t+1))) \\ & \leq \frac{\exp(-CN)}{(t+1)^{CN}}, \end{aligned} \quad (87)$$

and, thus,

$$\begin{aligned} \mathcal{P}(\bar{Q}_i \geq 5\lambda_{sup}CN(1 + \log(t+1))) & = \int \mathcal{P}(\bar{Q}_i \geq 5\lambda_{sup}CN(1 + \log(t+1)) | X_i) d\mathcal{P}_{X_i} \\ & \leq \frac{\exp(-CN)}{(t+1)^{CN}} \int d\mathcal{P}_{X_i} \equiv \frac{\exp(-CN)}{(t+1)^{CN}}. \end{aligned} \quad (88)$$

However, we would like to produce a bound on the supremum of all the $Q_i, i \in \mathbb{N}_t$. Indeed, using the naive union bound,

$$\mathcal{P}\left(\bigcup_{i \in \mathbb{N}_t} \{\bar{Q}_i \geq 5\lambda_{sup}CN(1 + \log(t+1))\}\right) \leq \sum_{i \in \mathbb{N}_t} \mathcal{P}(\bar{Q}_i \geq \lambda_{sup}5CN(1 + \log(t+1)))$$

$$\leq \frac{(t+1) \exp(-CN)}{(t+1)^{CN}} \equiv \frac{\exp(-CN)}{(t+1)^{CN-1}}$$

or, equivalently,

$$\begin{aligned} \mathcal{P} \left(\sup_{i \in \mathbb{N}_t} \bar{Q}_i < 5\lambda_{sup} CN (1 + \log(t+1)) \right) &\equiv \mathcal{P} \left(\{\bar{Q}_i < 5\lambda_{sup} CN (1 + \log(t+1)), \forall i \in \mathbb{N}_t\} \right) \\ &\equiv \mathcal{P} \left(\bigcap_{i \in \mathbb{N}_t} \{\bar{Q}_i < 5\lambda_{sup} CN (1 + \log(t+1))\} \right) \\ &\geq 1 - \frac{\exp(-CN)}{(t+1)^{CN-1}}, \end{aligned} \quad (89)$$

holding true $\forall t \in \mathbb{N}$. Consequently, working in the same fashion as above, it is true that, with at least the same probability of success,

$$\begin{aligned} \sup_{i \in \mathbb{N}_t} Q_i(\omega) &< 5\lambda_{sup} CN (1 + \log(t+1)) + 2\sqrt{5\lambda_{sup} CN (1 + \log(t+1))\mu_{sup} + \mu_{sup}^2} \\ &< 5\lambda_{sup} \left(1 + 2\mu_{sup} + \mu_{sup}^2\right) CN (1 + \log(t+1)) \end{aligned} \quad (90)$$

or, setting $\gamma_1 \triangleq 5\lambda_{sup} (1 + \mu_{sup})^2 > 1$,

$$\sup_{i \in \mathbb{N}_t} Q_i(\omega) < \gamma_1 CN (1 + \log(t+1)). \quad (91)$$

Now, under the alternative base measure $\tilde{\mathcal{P}}$, \mathbf{y}_t constitutes a Gaussian vector white noise process with zero mean and covariance matrix the identity, statistically independent of the process X_t (see Theorem 2). That is, for each t , the elements of \mathbf{y}_t are themselves independent to each other. Thus, for all $t \in \mathbb{N}$ and for all $i \in \mathbb{N}_t$ and using similar arguments as the ones made above, it should be true that

$$\tilde{\mathcal{P}}(Q_i < 5CN (1 + \log(t+1))) \geq 1 - \frac{\exp(-CN)}{(t+1)^{CN}} \quad (92)$$

and taking the union bound, we end up with the inequality

$$\tilde{\mathcal{P}} \left(\sup_{i \in \mathbb{N}_t} Q_i < 5CN (1 + \log(t+1)) \right) \geq 1 - \frac{\exp(-CN)}{(t+1)^{CN-1}}. \quad (93)$$

Defining $\gamma \triangleq \max\{\gamma_1, 5\} \equiv \gamma_1$, it must be true that, for all $t \in \mathbb{N}$,

$$\begin{aligned} \min \left\{ \mathcal{P} \left(\sup_{i \in \mathbb{N}_t} Q_i < \gamma CN (1 + \log(t+1)) \right), \tilde{\mathcal{P}} \left(\sup_{i \in \mathbb{N}_t} Q_i < \gamma CN (1 + \log(t+1)) \right) \right\} \\ \geq 1 - \frac{\exp(-CN)}{(t+1)^{CN-1}}, \end{aligned} \quad (94)$$

therefore completing the proof of the lemma. ■

Continuing our presentation of preliminary results towards the proof of Theorem (3) and leveraging the power of \mathcal{C} -weak convergence and Lemma 1, let us present the following lemma, connecting \mathcal{C} -weak convergence of random variables with convergence in the \mathcal{L}_1 sense.

Lemma 8. (From \mathcal{C} -Weak Convergence to Convergence in \mathcal{L}_1) Consider the sequence of discrete time stochastic processes $\left\{X_t^A\right\}_{A \in \mathbb{N}}$, as well as a “limit” process $X_t, t \in \mathbb{N}$, all being $(\mathbb{R}, \mathcal{S} \triangleq \mathcal{B}(\mathbb{R}))$ -valued and all defined on a common base space $(\Omega, \mathcal{F}, \mathcal{P})$. Further, suppose that all members of the collection $\left\{\left\{X_t^A\right\}_{A \in \mathbb{N}}, X_t\right\}_{t \in \mathbb{N}}$ are almost surely bounded in $\mathcal{Z} \equiv [a, b]$ (with $-\infty < a < b < \infty$) and that

$$\mathcal{P}_{X_t^A|X_t}^A(\cdot|X_t) \xrightarrow[A \rightarrow \infty]{\mathcal{W}} \delta_{X_t}(\cdot) \equiv \mathbb{1}_{(\cdot)}(X_t), \quad \forall t \in \mathbb{N}, \quad (95)$$

that is, the sequence $\left\{X_t^A\right\}_{A \in \mathbb{N}}$ is marginally \mathcal{C} -weakly convergent to X_t , given X_t , for all t . Then, it is true that

$$\mathbb{E} \left\{ \left| X_t - X_t^A \right| \right\} \xrightarrow[A \rightarrow \infty]{} 0, \quad \forall t \in \mathbb{N}, \quad (96)$$

or, equivalently, $X_t^A \xrightarrow[A \rightarrow \infty]{\mathcal{L}_1} X_t$, for all t .

Proof of Lemma 8. Let all the hypotheses of Lemma 8 hold true. Then, we know that, $\forall t \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \mathcal{P}_{X_t^A|X_t}^A(\mathcal{A}|X_t(\omega)) = \delta_{X_t(\omega)}(\mathcal{A}), \quad \mathcal{P} - a.e., \quad (97)$$

for all continuity Borel sets $\mathcal{A} \in \mathcal{S}$. Using the tower property, it is also true that

$$\mathbb{E} \left\{ \left| X_t - X_t^A \right| \right\} \equiv \mathbb{E} \left\{ \mathbb{E} \left\{ \left| X_t - X_t^A \right| \middle| \sigma \{X_t\} \right\} \right\}. \quad (98)$$

Therefore, in order to show that $X_t^A \xrightarrow[A \rightarrow \infty]{\mathcal{L}_1} X_t$ for each $t \in \mathbb{N}$, it suffices to show that

$$\mathbb{E} \left\{ \left| X_t - X_t^A \right| \middle| \sigma \{X_t\} \right\}(\omega) \xrightarrow[A \rightarrow \infty]{a.s.} 0, \quad \forall t \in \mathbb{N}. \quad (99)$$

Then, the Dominated Convergence Theorem would produce the desired result.

Of course, because all members of the collection $\left\{\left\{X_t^A\right\}_{A \in \mathbb{N}}, X_t\right\}_{t \in \mathbb{N}}$ are almost surely bounded in \mathcal{Z} , all members of the collection $\left\{\left\{\left| X_t - X_t^A \right| \right\}_{A \in \mathbb{N}}\right\}_{t \in \mathbb{N}}$ must be bounded almost surely in the compact set $\widehat{\mathcal{Z}} \triangleq [0, 2\delta] \subset \mathbb{R}$, where $\delta \triangleq \max\{|a|, |b|\}$.

Let us define the continuous and bounded function

$$f(x) \triangleq \begin{cases} x, & \text{if } x \in \widehat{\mathcal{Z}} \\ 2\delta, & \text{if } x > 2\delta \\ 0, & \text{if } x < 0 \end{cases}. \quad (100)$$

Then, from Lemma 1 and using conditional probability measures it must be true that for each $t \in \mathbb{N}$, a version of the conditional expectation of interest is explicitly given by

$$\mathbb{E} \left\{ f \left(\left| X_t - X_t^A \right| \right) \middle| \sigma \{X_t\} \right\}(\omega) \equiv \int f(|X_t(\omega) - x|) \mathcal{P}_{X_t^A|X_t}^A(dx|X_t(\omega)) \xrightarrow[A \rightarrow \infty]{} 0$$

$$\xrightarrow{\mathbf{A} \rightarrow \infty} \int f(|X_t(\omega) - x|) \delta_{X_t(\omega)}(dx) \equiv 0, \quad \mathcal{P} - a.e., \quad (101)$$

since, for each $\omega \in \Omega$, $X_t(\omega)$ is constant. Further, by definition of f ,

$$\begin{aligned} \mathbb{E} \left\{ f \left(\left| X_t - X_t^{\mathbf{A}} \right| \right) \middle| \sigma \{X_t\} \right\} (\omega) &\equiv \mathbb{E} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \mathbb{1}_{(|X_t - X_t^{\mathbf{A}}|)} \left(\widehat{\mathcal{Z}} \right) \middle| \sigma \{X_t\} \right\} (\omega) \\ &\equiv \mathbb{E} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \middle| \sigma \{X_t\} \right\} (\omega), \quad \mathcal{P} - a.e., \end{aligned} \quad (102)$$

and for all $t \in \mathbb{N}$, which means that

$$\mathbb{E} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \middle| \sigma \{X_t\} \right\} (\omega) \xrightarrow[\mathbf{A} \rightarrow \infty]{a.s.} 0, \quad \forall t \in \mathbb{N}. \quad (103)$$

Calling dominated convergence proves the result. \blacksquare

Additionally, the following useful (to us) result is also true. The proof, being elementary, is omitted.

Lemma 9. (Convergence of the Supremum) *Pick any natural $T < \infty$. If, under any circumstances,*

$$\mathbb{E} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \right\} \xrightarrow[\mathbf{A} \rightarrow \infty]{} 0, \quad \forall t \in \mathbb{N}_T, \quad (104)$$

then

$$\sup_{t \in \mathbb{N}_T} \mathbb{E} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \right\} \xrightarrow[\mathbf{A} \rightarrow \infty]{} 0. \quad (105)$$

4.3 The Key Lemma

We are now ready to present our key lemma, which will play an important role in establishing our main result (Theorem 3) later on. For proving this result, we make use of all the intermediate ones presented in the previous subsections.

Lemma 10. (Convergence of the Likelihoods) *Consider the stochastic process*

$$\widehat{\Lambda}_t \triangleq \frac{\exp \left(-\frac{1}{2} \sum_{i \in \mathbb{N}_t} \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) \right)}{\prod_{i \in \mathbb{N}_t} \sqrt{\det(\mathbf{C}_i(X_i))}} \triangleq \frac{\mathfrak{N}_t}{\mathfrak{D}_t}, \quad t \in \mathbb{N}. \quad (106)$$

Consider also the process $\widehat{\Lambda}_t^{\mathbf{A}} \triangleq \mathfrak{N}_t^{\mathbf{A}} / \mathfrak{D}_t^{\mathbf{A}}$, defined exactly the same way as $\widehat{\Lambda}_t$, but replacing X_i with the approximation $X_i^{\mathbf{A}}$, $\forall i \in \mathbb{N}_t$. Further, pick any natural $T < \infty$ and suppose either of the following:

- For all $t \in \mathbb{N}_T$, the sequence $\left\{ X_t^{\mathbf{A}} \right\}_{\mathbf{A} \in \mathbb{N}}$ is marginally \mathcal{C} -weakly convergent to X_t , given X_t , that is,

$$\mathcal{P}_{X_t^{\mathbf{A}} | X_t}^{\mathbf{A}}(\cdot | X_t) \xrightarrow[\mathbf{A} \rightarrow \infty]{\mathcal{W}} \delta_{X_t}(\cdot), \quad \forall t \in \mathbb{N}_T. \quad (107)$$

- For all $t \in \mathbb{N}_T$, the sequence $\left\{ X_t^{\mathbf{A}} \right\}_{\mathbf{A} \in \mathbb{N}}$ is marginally convergent to X_t in probability, that is,

$$X_t^{\mathbf{A}} \xrightarrow[\mathbf{A} \rightarrow \infty]{\mathcal{P}} X_t, \quad \forall t \in \mathbb{N}_T. \quad (108)$$

Then, there exists a measurable subset $\widehat{\Omega}_T \subseteq \Omega$, such that

$$\lim_{A \rightarrow \infty} \sup_{t \in \mathbb{N}_T} \sup_{\omega \in \widehat{\Omega}_T} \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| \middle| \mathcal{D}_t \right\} (\omega) \equiv 0, \quad (109)$$

where the $\mathcal{P}, \widetilde{\mathcal{P}}$ -measures of $\widehat{\Omega}_T$ satisfy

$$\min \left\{ \mathcal{P} \left(\widehat{\Omega}_T \right), \widetilde{\mathcal{P}} \left(\widehat{\Omega}_T \right) \right\} \geq 1 - \frac{\exp(-CN)}{(T+1)^{CN-1}}, \quad (110)$$

for any free but finite constant $C \geq 1$.

Proof of Lemma 10. From Lemma 2, it is true that

$$\begin{aligned} \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| &\equiv \left| \frac{\mathfrak{N}_t}{\mathfrak{D}_t} - \frac{\mathfrak{N}_t^A}{\mathfrak{D}_t^A} \right| \leq \frac{|\mathfrak{N}_t - \mathfrak{N}_t^A|}{|\mathfrak{D}_t^A|} + |\mathfrak{N}_t| \left| \frac{1}{\mathfrak{D}_t} - \frac{1}{\mathfrak{D}_t^A} \right| \\ &\leq \frac{|\mathfrak{N}_t - \mathfrak{N}_t^A|}{|\mathfrak{D}_t^A|} + \left| \frac{1}{\mathfrak{D}_t} - \frac{1}{\mathfrak{D}_t^A} \right|. \end{aligned} \quad (111)$$

We first concentrate on the determinant part (second term) of the RHS of (111). Directly invoking Lemma 3, it will be true that

$$\begin{aligned} &\left| \frac{1}{\mathfrak{D}_t} - \frac{1}{\mathfrak{D}_t^A} \right| \\ &\equiv \left| \prod_{i=0}^t \frac{1}{\sqrt{\det(\mathbf{C}_i(X_i))}} - \prod_{i=0}^t \frac{1}{\sqrt{\det(\mathbf{C}_i(X_i^A))}} \right| \\ &\leq \sum_{i=0}^t \left(\prod_{j=0}^{i-1} \frac{1}{\sqrt{\det(\mathbf{C}_j(X_j))}} \right) \left(\prod_{j=i+1}^t \frac{1}{\sqrt{\det(\mathbf{C}_j(X_j^A))}} \right) \frac{|\sqrt{\det(\mathbf{C}_i(X_i))} - \sqrt{\det(\mathbf{C}_i(X_i^A))}|}{\sqrt{\det(\mathbf{C}_i(X_i)) \det(\mathbf{C}_i(X_i^A))}} \\ &= \sum_{i=0}^t \left(\prod_{j=0}^{i-1} \frac{1}{\sqrt{\prod_{n=1}^N \lambda_n(\mathbf{C}_j(X_j))}} \right) \left(\prod_{j=i+1}^t \frac{1}{\sqrt{\prod_{n=1}^N \lambda_n(\mathbf{C}_j(X_j^A))}} \right) \frac{|\sqrt{\det(\mathbf{C}_i(X_i))} - \sqrt{\det(\mathbf{C}_i(X_i^A))}|}{\sqrt{\prod_{n=1}^N \lambda_n(\mathbf{C}_j(X_j)) \lambda_n(\mathbf{C}_j(X_j^A))}} \\ &\leq \sum_{i=0}^t \frac{1}{2\lambda_{inf}^{Ni/2} \lambda_{inf}^{N(t-i)/2}} \frac{|\det(\mathbf{C}_i(X_i)) - \det(\mathbf{C}_i(X_i^A))|}{\lambda_{inf}^N} \\ &\equiv \frac{1}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \sum_{i=0}^t |\det(\mathbf{C}_i(X_i)) - \det(\mathbf{C}_i(X_i^A))|. \end{aligned} \quad (112)$$

From Lemma 5, we can bound the RHS of the above expression as

$$\begin{aligned} \left| \frac{1}{\mathfrak{D}_t} - \frac{1}{\mathfrak{D}_t^A} \right| &\leq \frac{NK_{DET}}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \sum_{i=0}^t \left\| \mathbf{C}_i(X_i) - \mathbf{C}_i(X_i^A) \right\|_F \\ &\equiv \frac{NK_{DET}}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \sum_{i=0}^t \left\| \boldsymbol{\Sigma}_i(X_i) - \boldsymbol{\Sigma}_i(X_i^A) \right\|_F. \end{aligned} \quad (113)$$

And from Lemma 4, (113) becomes

$$\left| \frac{1}{\mathfrak{D}_t} - \frac{1}{\mathfrak{D}_t^A} \right| \leq \frac{N^2 K_{DET} K_{\boldsymbol{\Sigma}}}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \sum_{i=0}^t |X_i - X_i^A|. \quad (114)$$

We now turn our attention to the ‘‘difference of exponentials’’ part (first term) of the RHS of (111). First, we know that

$$\prod_{i=0}^t \det(\mathbf{C}_i(X_i^A)) \geq \prod_{i=0}^t \prod_{j=1}^N \lambda_{inf} \equiv \lambda_{inf}^{N(t+1)}, \quad (115)$$

yielding the inequality

$$\frac{|\mathfrak{N}_t - \mathfrak{N}_t^A|}{|\mathfrak{D}_t^A|} \leq \frac{|\mathfrak{N}_t - \mathfrak{N}_t^A|}{\sqrt{\lambda_{inf}^{N(t+1)}}}, \quad (116)$$

where $\lambda_{inf} > 1$ (see Assumption 1). Next, making use of the inequality [11]

$$|\exp(\alpha) - \exp(\beta)| \leq |\alpha - \beta| (\exp(\alpha) + \exp(\beta)), \quad (117)$$

$\forall (\alpha, \beta) \in \mathbb{R}^2$, the absolute difference on the numerator of (116) can be upper bounded as

$$\begin{aligned} |\mathfrak{N}_t - \mathfrak{N}_t^A| &\leq \frac{1}{2} \left| \sum_{i=0}^t \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i^T(X_i^A) \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right| \left(\mathfrak{N}_t + \mathfrak{N}_t^A \right) \\ &\leq \sum_{i=0}^t \left| \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i^T(X_i^A) \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right|. \end{aligned} \quad (118)$$

Concentrating on each member of the series above in the last line of (118) and calling Lemma 2, it is true that

$$\begin{aligned} &\left| \bar{\mathbf{y}}_i^T(X_i) \left[\mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) \right] - \bar{\mathbf{y}}_i^T(X_i^A) \left[\mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right] \right| \\ &\leq \|\bar{\mathbf{y}}_i(X_i)\|_2 \left\| \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right\|_2 + \\ &\quad + \left\| \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right\|_2 \left\| \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A) \right\|_2. \end{aligned} \quad (119)$$

Calling Lemma 2 again for the term multiplying the quantity $\|\bar{\mathbf{y}}_i(X_i)\|_2$ in the RHS of the above expression, we arrive at the inequalities

$$\left| \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i^T(X_i^A) \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right|$$

$$\begin{aligned}
&\leq \|\bar{\mathbf{y}}_i(X_i)\|_2 \|\mathbf{C}_i^{-1}(X_i)\|_2 \|\bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A)\|_2 \\
&\quad + \|\bar{\mathbf{y}}_i(X_i^A)\|_2 \|\mathbf{C}_i^{-1}(X_i^A)\|_2 \|\bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A)\|_2 \\
&\quad + \|\bar{\mathbf{y}}_i(X_i)\|_2 \|\bar{\mathbf{y}}_i(X_i^A)\|_2 \|\mathbf{C}_i^{-1}(X_i) - \mathbf{C}_i^{-1}(X_i^A)\|_2 \\
&\leq \frac{\|\bar{\mathbf{y}}_i(X_i)\|_2}{\lambda_{inf}} \|\bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A)\|_2 + \frac{\|\bar{\mathbf{y}}_i(X_i^A)\|_2}{\lambda_{inf}} \|\bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A)\|_2 \\
&\quad + \|\bar{\mathbf{y}}_i(X_i)\|_2 \|\bar{\mathbf{y}}_i(X_i^A)\|_2 \|\mathbf{C}_i^{-1}(X_i) - \mathbf{C}_i^{-1}(X_i^A)\|_2, \tag{120}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\left| \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i^T(X_i^A) \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right| \\
&\leq \frac{\|\bar{\mathbf{y}}_i(X_i)\|_2 + \|\bar{\mathbf{y}}_i(X_i^A)\|_2}{\lambda_{inf}} \|\bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i(X_i^A)\|_2 + \\
&\quad + \|\bar{\mathbf{y}}_i(X_i)\|_2 \|\bar{\mathbf{y}}_i(X_i^A)\|_2 \|\mathbf{C}_i^{-1}(X_i) - \mathbf{C}_i^{-1}(X_i^A)\|_2. \tag{121}
\end{aligned}$$

Now, recalling Assumption 2, the definition of $\bar{\mathbf{y}}_i(X_i)$ (resp. for X_i^A) and invoking Lemma 6, it must be true that

$$\begin{aligned}
&\left| \bar{\mathbf{y}}_i^T(X_i) \mathbf{C}_i^{-1}(X_i) \bar{\mathbf{y}}_i(X_i) - \bar{\mathbf{y}}_i^T(X_i^A) \mathbf{C}_i^{-1}(X_i^A) \bar{\mathbf{y}}_i(X_i^A) \right| \\
&\leq \frac{\|\mathbf{y}_i - \boldsymbol{\mu}_i(X_i)\|_2 + \|\mathbf{y}_i - \boldsymbol{\mu}_i(X_i^A)\|_2}{\lambda_{inf}} \|\boldsymbol{\mu}_i(X_i) - \boldsymbol{\mu}_i(X_i^A)\|_2 \\
&\quad + \|\mathbf{y}_i - \boldsymbol{\mu}_i(X_i)\|_2 \|\mathbf{y}_i - \boldsymbol{\mu}_i(X_i^A)\|_2 \|\mathbf{C}_i^{-1}(X_i) - \mathbf{C}_i^{-1}(X_i^A)\|_2 \\
&\leq \left(K_\mu \frac{2(\|\mathbf{y}_i\|_2 + \mu_{sup})}{\lambda_{inf}} + K_{INV} (\|\mathbf{y}_i\|_2 + \mu_{sup})^2 \right) |X_i - X_i^A| \triangleq \Theta(\mathbf{y}_i) |X_i - X_i^A|. \tag{122}
\end{aligned}$$

Using the above inequality, the RHS of (116) can be further bounded from above as

$$\frac{|\mathfrak{N}_t - \mathfrak{N}_t^A|}{|\mathfrak{D}_t^A|} \leq \frac{\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)}{\sqrt{\lambda_{inf}^{N(t+1)}}} \sum_{i=0}^t |X_i - X_i^A|. \tag{123}$$

Therefore, we can bound the RHS of (111) as

$$\left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| \leq \left(\frac{\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)}{\sqrt{\lambda_{inf}^{N(t+1)}}} + \frac{N^2 K_{DET} K_\Sigma}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \right) \sum_{i=0}^t |X_i - X_i^A|. \tag{124}$$

Taking conditional expectations on both sides of (124), observing that the quantity $\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)$ constitutes a $\{\mathcal{Y}_t\}$ -adapted process and recalling that under the base measure $\tilde{\mathcal{P}}$ (see Theorem 2),

the processes \mathbf{y}_t and X_t (resp. X_t^A) are statistically independent, we can write

$$\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \hat{\Lambda}_t - \hat{\Lambda}_t^A \right| \middle| \mathcal{Y}_t \right\} \leq \left(\frac{\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)}{\sqrt{\lambda_{inf}^{N(t+1)}}} + \frac{N^2 K_{DET} K_{\Sigma}}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \right) \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \sum_{i=0}^t |X_i - X_i^A| \right\}, \quad (125)$$

$\tilde{\mathcal{P}} - a.e.$, and, because $\mathcal{P} \ll_{\mathcal{H}_t} \tilde{\mathcal{P}}$, $\mathcal{P} - a.e.$ as well. From the last inequality, we can readily observe that in order to be able to talk about any kind of uniform convergence regarding the RHS, it is vital to ensure that the random variable $\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)$ is bounded from above. However, because the support of $\|\mathbf{y}_i\|_2$ is infinite, it is impossible to bound $\sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i)$ in the almost sure sense. Nevertheless, Lemma 7 immediately implies that there exists a measurable subset $\hat{\Omega}_\tau \subseteq \Omega$ with

$$\min \left\{ \mathcal{P}(\hat{\Omega}_\tau), \tilde{\mathcal{P}}(\hat{\Omega}_\tau) \right\} \geq 1 - \frac{\exp(-CN)}{(\tau+1)^{CN-1}} \quad (126)$$

such that, for all $\omega \in \hat{\Omega}_\tau$,

$$\sup_{i \in \mathbb{N}_\tau} \|\mathbf{y}_i(\omega)\|_2^2 \equiv \sup_{i \in \mathbb{N}_\tau} \|\mathbf{y}_i\|_2^2 < \gamma CN (1 + \log(1 + \tau)), \quad (127)$$

for some fixed constant $\gamma > 1$, for any $C \geq 1$ and for any *fixed* $\tau \in \mathbb{N}$. Choosing $\tau \equiv T < \infty$, it is true that

$$\begin{aligned} \sup_{i \in \mathbb{N}_t} \Theta(\mathbf{y}_i) &\leq \sup_{i \in \mathbb{N}_T} \Theta(\mathbf{y}_i) \\ &\leq \sup_{i \in \mathbb{N}_T} \left[K_\mu \frac{2(\|\mathbf{y}_i\|_2 + \mu_{sup})}{\lambda_{inf}} + K_{INV} (\|\mathbf{y}_i\|_2 + \mu_{sup})^2 \right] \\ &< \left(K_\mu \frac{2\tilde{\gamma}}{\lambda_{inf}} + K_{INV} \tilde{\gamma}^2 \right) CN (1 + \log(1 + T)) \\ &\triangleq K_o CN (1 + \log(1 + T)), \quad \forall t \in \mathbb{N}_T, \end{aligned} \quad (128)$$

where $\tilde{\gamma} \triangleq \sqrt{\gamma} + \mu_{sup}$. Therefore, it will be true that

$$\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \hat{\Lambda}_t - \hat{\Lambda}_t^A \right| \middle| \mathcal{Y}_t \right\} \leq \left(\frac{K_o CN (1 + \log(1 + T))}{\sqrt{\lambda_{inf}^{N(t+1)}}} + \frac{N^2 K_{DET} K_{\Sigma}}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \right) \sum_{i=0}^t \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ |X_i - X_i^A| \right\}, \quad (129)$$

for all $t \in \mathbb{N}_T$, with probability at least

$$1 - \frac{\exp(-CN)}{(T+1)^{CN-1}},$$

under either \mathcal{P} or $\tilde{\mathcal{P}}$. Further,

$$\sum_{i=0}^t \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ |X_i - X_i^A| \right\} \leq (t+1) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ |X_\tau - X_\tau^A| \right\}. \quad (130)$$

Then, with the same probability of success,

$$\begin{aligned}
& \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\} \\
& \leq \left(\frac{K_o C N (1 + \log(1 + T)) (t + 1)}{\sqrt{\lambda_{inf}^{N(t+1)}}} + \frac{K_{DET} K_{\Sigma} N^2 (t + 1)}{2\sqrt{\lambda_{inf}^{N(t+2)}}} \right) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\} \\
& \leq \left(\frac{K_o C N (1 + \log(1 + T)) (T + 1)}{\lambda_{inf}^{N/2}} + \frac{K_{DET} K_{\Sigma} N^2 (T + 1)}{2\lambda_{inf}^N} \right) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\} \\
& \triangleq K_G(T) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\}, \quad \forall t \in \mathbb{N}_T, \tag{131}
\end{aligned}$$

where $K_G(T) \equiv \mathcal{O}(T \log(T))$. *Alternatively*, upper bounding the functions comprised by the quantities t, N, λ_{inf} in the second and third lines of the expressions above as (note that, obviously, $t + 1 \geq 1, \forall t \in \mathbb{R}_+$)

$$\frac{N(t+1)}{\sqrt{\lambda_{inf}^{N(t+1)}}} \leq \max_{t \in \mathbb{R}_+} \frac{N(t+1)^2}{\sqrt{\lambda_{inf}^{N(t+1)}}} \quad \text{and} \tag{132}$$

$$\frac{N^2(t+1)}{\sqrt{\lambda_{inf}^{N(t+2)}}} \leq \max_{t \in \mathbb{R}_+} \frac{N^2(t+1)^2}{\sqrt{\lambda_{inf}^{N(t+2)}}}, \tag{133}$$

respectively, we can also define

$$K_G(T) \triangleq \frac{16K_o C (1 + \log(1 + T)) \lambda_{inf}^{-2/\log(\lambda_{inf})}}{N (\log(\lambda_{inf}))^2} + \frac{8K_{DET} K_{\Sigma} \lambda_{inf}^{-N} \lambda_{inf}^{-2/\log(\lambda_{inf})}}{(\log(\lambda_{inf}))^2}, \tag{134}$$

where, in this case, $K_G(T) \equiv \mathcal{O}(\log(T))$. Note, however, that although its dependence on T is logarithmic, $K_G(T)$ may still be large due to the inability to compensate for the size of K_o . In any case, for all $\omega \in \widehat{\Omega}_T$,

$$\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\} (\omega) \leq K_G(T) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\}, \quad \forall t \in \mathbb{N}_T, \tag{135}$$

Therefore, we get

$$\sup_{\omega \in \widehat{\Omega}_T} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\} (\omega) \leq K_G(T) \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\}, \quad \forall t \in \mathbb{N}_T \tag{136}$$

and further taking the supremum over $t \in \mathbb{N}_T$ on both sides, it must be true that

$$\begin{aligned}
\sup_{t \in \mathbb{N}_T} \sup_{\omega \in \widehat{\Omega}_T} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\} (\omega) & \leq K_G(T) \sup_{t \in \mathbb{N}_T} \sup_{\tau \in \mathbb{N}_t} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_{\tau} - X_{\tau}^{\mathbf{A}} \right| \right\} \\
& \equiv K_G(T) \sup_{t \in \mathbb{N}_T} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \right\}. \tag{137}
\end{aligned}$$

Finally, if either

$$\mathcal{P}_{X_t^A|X_t}^A(\cdot|X_t) \xrightarrow[A \rightarrow \infty]{\mathcal{W}} \delta_{X_t}(\cdot) \equiv \mathbf{1}_{(\cdot)}(X_t), \quad \forall t \in \mathbb{N}_T, \quad (138)$$

or

$$X_t^A \xrightarrow[A \rightarrow \infty]{\mathcal{P}} X_t, \quad \forall t \in \mathbb{N}_T \quad (139)$$

and given that since the members of $\left\{X_t^A\right\}_{A \in \mathbb{N}}$ are almost surely bounded in \mathcal{Z} , the aforementioned sequence is also uniformly integrable for all $t \in \mathbb{N}$, it must be true that (see Lemma 8)

$$\mathbb{E}_{\tilde{\mathcal{P}}}\left\{\left|X_t - X_t^A\right|\right\} \xrightarrow[A \rightarrow \infty]{} 0, \quad \forall t \in \mathbb{N}_T. \quad (140)$$

Then, Lemma 9 implies that

$$\sup_{t \in \mathbb{N}_T} \mathbb{E}_{\tilde{\mathcal{P}}}\left\{\left|X_t - X_t^A\right|\right\} \xrightarrow[A \rightarrow \infty]{} 0, \quad (141)$$

which in turn implies the existence of the limit on the LHS of (137). QED. ■

4.4 Finishing the Proof of Theorem 3

Considering the absolute difference of the RHSs of (44) and (43), it is true that (see Lemma 2)

$$\left| \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \Lambda_t | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\Lambda_t | \mathcal{Y}_t\}} - \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t^A \Lambda_t^A | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\Lambda_t^A | \mathcal{Y}_t\}} \right| \equiv \left| \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \widehat{\Lambda}_t | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t | \mathcal{Y}_t\}} - \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t^A \widehat{\Lambda}_t^A | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\}} \right|, \quad (142)$$

due to the fact that the increasing stochastic process

$$\prod_{i \in \mathbb{N}_t} \exp\left(\frac{1}{2} \|\mathbf{y}_i\|_2^2\right) \equiv \exp\left(\frac{1}{2} \sum_{i \in \mathbb{N}_t} \|\mathbf{y}_i\|_2^2\right) \quad (143)$$

is $\{\mathcal{Y}_t\}$ -adapted. Then, we can write

$$\begin{aligned} & \left| \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \widehat{\Lambda}_t | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t | \mathcal{Y}_t\}} - \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t^A \widehat{\Lambda}_t^A | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\}} \right| \\ & \equiv \frac{\left| \mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \widehat{\Lambda}_t | \mathcal{Y}_t\} \mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\} - \mathbb{E}_{\tilde{\mathcal{P}}}\{X_t^A \widehat{\Lambda}_t^A | \mathcal{Y}_t\} \mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t | \mathcal{Y}_t\} \right|}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t | \mathcal{Y}_t\} \mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\}} \\ & \leq \frac{\left| \mathbb{E}_{\tilde{\mathcal{P}}}\{X_t | \mathcal{Y}_t\} \left| \mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t - \widehat{\Lambda}_t^A | \mathcal{Y}_t\} \right| \right|}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\}} + \frac{\left| \mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \widehat{\Lambda}_t - X_t^A \widehat{\Lambda}_t^A | \mathcal{Y}_t\} \right|}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\widehat{\Lambda}_t^A | \mathcal{Y}_t\}}, \quad \tilde{\mathcal{P}}, \mathcal{P} - a.e.. \quad (144) \end{aligned}$$

Let us first focus on the difference on the numerator of the second ratio of the RHS of (144). Recalling that $\delta \equiv \max\{|a|, |b|\}$, we can then write

$$\left| \mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \widehat{\Lambda}_t - X_t^A \widehat{\Lambda}_t^A | \mathcal{Y}_t\} \right| \leq \mathbb{E}_{\tilde{\mathcal{P}}}\left\{\left|X_t \widehat{\Lambda}_t - X_t^A \widehat{\Lambda}_t^A\right| \middle| \mathcal{Y}_t\right\}$$

$$\begin{aligned}
&\leq \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ |X_t| \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| + \left| \widehat{\Lambda}_t^A \right| \left| X_t - X_t^A \right| \middle| \mathcal{Y}_t \right\} \\
&\leq \delta \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| \middle| \mathcal{Y}_t \right\} + \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_t - X_t^A \right| \middle| \mathcal{Y}_t \right\} \\
&\equiv \delta \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^A \right| \middle| \mathcal{Y}_t \right\} + \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \left| X_t - X_t^A \right| \right\}, \tag{145}
\end{aligned}$$

On the other hand, for the denominator for (144), it is true that

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \widehat{\Lambda}_t^A \middle| \mathcal{Y}_t \right\} &\equiv \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \frac{\exp \left(-\frac{1}{2} \sum_{i \in \mathbb{N}_t} \bar{\mathbf{y}}_i^T (X_i^A) \mathbf{C}_i^{-1} (X_i^A) \bar{\mathbf{y}}_i (X_i^A) \right)}{\prod_{i \in \mathbb{N}_t} \sqrt{\det (\mathbf{C}_i (X_i^A))}} \middle| \mathcal{Y}_t \right\} \\
&\geq \frac{\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \exp \left(-\frac{1}{2} \sum_{i \in \mathbb{N}_t} \bar{\mathbf{y}}_i^T (X_i^A) \mathbf{C}_i^{-1} (X_i^A) \bar{\mathbf{y}}_i (X_i^A) \right) \middle| \mathcal{Y}_t \right\}}{\sqrt{\lambda_{sup}^{N(t+1)}}} \\
&\geq \frac{\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \exp \left(-\frac{1}{2\lambda_{inf}} \sum_{i \in \mathbb{N}_t} (\|\mathbf{y}_i\|_2 + \mu_{sup})^2 \right) \middle| \mathcal{Y}_t \right\}}{\sqrt{\lambda_{sup}^{N(t+1)}}} \\
&\equiv \frac{\exp \left(-\frac{1}{2\lambda_{inf}} \sum_{i \in \mathbb{N}_t} (\|\mathbf{y}_i\|_2 + \mu_{sup})^2 \right)}{\sqrt{\lambda_{sup}^{N(t+1)}}}, \quad \tilde{\mathcal{P}}, \mathcal{P} - a.e., \tag{146}
\end{aligned}$$

since the process $\sum_{i \in \mathbb{N}_t} (\|\mathbf{y}_i\|_2 + \mu_{sup})^2$ is $\{\mathcal{Y}_t\}$ -adapted. Now, from Lemma 7, we know that

$$\sup_{i \in \mathbb{N}_t} \|\mathbf{y}_i\|_2^2 \leq \sup_{i \in \mathbb{N}_T} \|\mathbf{y}_i\|_2^2 < \gamma CN (1 + \log (T + 1)), \quad \forall t \in \mathbb{N}_T, \tag{147}$$

where the last inequality holds with probability at least $1 - (T + 1)^{1-CN} \exp(-CN)$, under both base measures \mathcal{P} and $\tilde{\mathcal{P}}$, for any finite constant $C \geq 1$. Therefore, it can be trivially shown that

$$\mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \widehat{\Lambda}_t^A \middle| \mathcal{Y}_t \right\} \geq \frac{\exp \left(-\frac{\left(\sqrt{\gamma CN (1 + \log (T + 1))} + \mu_{sup} \right)^2 (T + 1)}{2\lambda_{inf}} \right)}{\sqrt{\lambda_{sup}^{N(t+1)}}} > 0, \quad \forall t \in \mathbb{N}_T, \tag{148}$$

implying that

$$\inf_{t \in \mathbb{N}_T} \inf_{\omega \in \widehat{\Omega}_T} \inf_{A \in \mathbb{N}} \mathbb{E}_{\tilde{\mathcal{P}}} \left\{ \widehat{\Lambda}_t^A \middle| \mathcal{Y}_t \right\} (\omega) > 0, \tag{149}$$

where $\widehat{\Omega}_T$ coincides with the event

$$\left\{ \omega \in \Omega \left| \sup_{i \in \mathbb{N}_t} \|\mathbf{y}_i\|_2^2 < \gamma CN (1 + \log(T+1)), \forall t \in \mathbb{N}_T \right. \right\}$$

with $\mathcal{P}, \widetilde{\mathcal{P}}$ -measure at least $1 - (T+1)^{1-CN} \exp(-CN)$. Of course, the existence of $\widehat{\Omega}_T$ follows from Lemma 7. Putting it altogether, (144) becomes (recall that the base measures \mathcal{P} and $\widetilde{\mathcal{P}}$ are equivalent)

$$\left| \mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\} - \mathcal{E}^{\mathbf{A}}(X_t | \mathcal{Y}_t) \right| \leq \frac{2\delta \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\} + \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \right\}}{\inf_{\mathbf{A} \in \mathbb{N}} \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \widehat{\Lambda}_t^{\mathbf{A}} \middle| \mathcal{Y}_t \right\}}, \quad \mathcal{P}, \widetilde{\mathcal{P}} - a.e., \quad (150)$$

where $\delta \equiv \max\{|a|, |b|\}$. Taking the supremum both with respect to $\omega \in \widehat{\Omega}_T$ and $t \in \mathbb{N}_T$ on both sides, we get

$$\begin{aligned} & \sup_{t \in \mathbb{N}_T} \sup_{\omega \in \widehat{\Omega}_T} \left| \mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\}(\omega) - \mathcal{E}^{\mathbf{A}}(X_t | \mathcal{Y}_t)(\omega) \right| \\ & \leq \frac{\sup_{t \in \mathbb{N}_T} \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \left| X_t - X_t^{\mathbf{A}} \right| \right\} + 2\delta \sup_{t \in \mathbb{N}_T} \sup_{\omega \in \widehat{\Omega}_T} \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \left| \widehat{\Lambda}_t - \widehat{\Lambda}_t^{\mathbf{A}} \right| \middle| \mathcal{Y}_t \right\}(\omega)}{\inf_{t \in \mathbb{N}_T} \inf_{\omega \in \widehat{\Omega}_T} \inf_{\mathbf{A} \in \mathbb{N}} \mathbb{E}_{\widetilde{\mathcal{P}}} \left\{ \widehat{\Lambda}_t^{\mathbf{A}} \middle| \mathcal{Y}_t \right\}(\omega)}, \end{aligned} \quad (151)$$

with

$$\min \left\{ \mathcal{P} \left(\widehat{\Omega}_T \right), \widetilde{\mathcal{P}} \left(\widehat{\Omega}_T \right) \right\} \geq 1 - \frac{\exp(-CN)}{(T+1)^{CN-1}}. \quad (152)$$

Finally, calling Lemma 10 and Lemma 9, and since the denominators of the fractions appearing in (151) are nonzero, its RHS tends to zero as $\mathbf{A} \rightarrow \infty$, under the respective hypotheses. Consequently, the LHS will also converge, therefore completing the proof of the Theorem 3. \blacksquare

5 Conclusion

In this paper, we have provided sufficient conditions for convergence of approximate, asymptotically optimal nonlinear filtering operators, for a general class of hidden stochastic processes, observed in a conditionally Gaussian noisy environment. In particular, employing a common change of measure argument, we have shown that using the same measurements, but replacing the “true” state by an approximation process, which converges to the former either in probability or in the \mathcal{C} -weak sense, one can define an approximate filtering operator, which converges to the optimal filter compactly in time and uniformly in an event occurring with probability nearly 1, at the same time constituting a purely quantitative justification of Egoroff’s theorem for the problem of interest. The results presented in this paper essentially provide a framework for analyzing the convergence properties of various classes of approximate nonlinear filters (either recursive or nonrecursive), such as existing grid based approaches, which are known to perform well in various applications.

Appendix: Proof of Theorem 1

The proof is astonishingly simple. Let the hypotheses of the statement of Theorem 1 hold true. To avoid useless notational congestion, let us also make the identifications

$$\mathcal{X}_t \triangleq \{X_i\}_{i \in \mathbb{N}_t} \quad \text{and} \quad \mathcal{Y}_t \triangleq \{Y_i\}_{i \in \mathbb{N}_t}. \quad (153)$$

Now, by definition of the conditional expectation operator and since we have assumed the existence of densities, it is true that

$$\begin{aligned} \widehat{X}_t &\equiv \mathbb{E}_{\mathcal{P}} \{X_t | Y_0, Y_1, \dots, Y_t\} \\ &= \int x_t f_{X_t | \mathcal{Y}_t}(x_t | \mathcal{Y}_t) dx_t \\ &= \frac{\int x_t f_{(X_t, \mathcal{Y}_t)}(x_t, \mathcal{Y}_t) dx_t}{f_{\mathcal{Y}_t}(\mathcal{Y}_t)} \\ &\equiv \frac{\int x_t f_t(x_0, x_1, \dots, x_t, \mathcal{Y}_t) \prod_{i=0}^t dx_i}{\int f_t(x_0, x_1, \dots, x_t, \mathcal{Y}_t) \prod_{i=0}^t dx_i} \\ &\equiv \frac{\int x_t \lambda_t \widetilde{f}_t(\{x_i\}_{i \in \mathbb{N}_t}, \mathcal{Y}_t) \prod_{i=0}^t dx_i}{\int \lambda_t \widetilde{f}_t(\{x_i\}_{i \in \mathbb{N}_t}, \mathcal{Y}_t) \prod_{i=0}^t dx_i}, \end{aligned} \quad (154)$$

where

$$\begin{aligned} \lambda_t &\triangleq \frac{f_t(x_0, x_1, \dots, x_t, Y_0(\omega), Y_1(\omega), \dots, Y_t(\omega))}{\widetilde{f}_t(x_0, x_1, \dots, x_t, Y_0(\omega), Y_1(\omega), \dots, Y_t(\omega))} \\ &\equiv \lambda_t(\{x_i\}_{i \in \mathbb{N}_t}, \mathcal{Y}_t(\omega)) \in \mathbb{R}_+, \quad \forall \omega \in \Omega, \end{aligned} \quad (155)$$

constitutes a “half ordinary function - half random variable” likelihood ratio and where the condition (16) ensures its boundedness. Of course, although the likelihood ratio can be indeterminate when both densities are zero, the respective points do not contribute in the computation of the relevant integrals presented above, because these belong to measurable sets corresponding to events of measure zero. From (154) and by definition of the conditional density of $\{X_i\}_{i \in \mathbb{N}_t}$ given $\{Y_i\}_{i \in \mathbb{N}_t}$, we immediately get

$$\begin{aligned} \widehat{X}_t &\equiv \mathbb{E}_{\mathcal{P}} \{X_t | \mathcal{Y}_t\} \\ &= \frac{\int x_t \lambda_t \widetilde{f}_{X_t | \mathcal{Y}_t}(\{x_i\}_{i \in \mathbb{N}_t} | \mathcal{Y}_t) \prod_{i=0}^t dx_i}{\int \lambda_t \widetilde{f}_{X_t | \mathcal{Y}_t}(\{x_i\}_{i \in \mathbb{N}_t} | \mathcal{Y}_t) \prod_{i=0}^t dx_i} \end{aligned}$$

$$\begin{aligned}
&= \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \Lambda_t | Y_0, Y_1, \dots, Y_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\Lambda_t | Y_0, Y_1, \dots, Y_t\}} \\
&\equiv \frac{\mathbb{E}_{\tilde{\mathcal{P}}}\{X_t \Lambda_t | \mathcal{Y}_t\}}{\mathbb{E}_{\tilde{\mathcal{P}}}\{\Lambda_t | \mathcal{Y}_t\}}, \tag{156}
\end{aligned}$$

which constitutes what we were initially set to show. ■

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